Advanced Acoustics

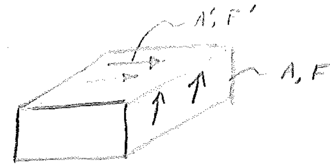
R.J. Marks II Class Notes

Rose-Hulman Institute of Technology (1973)

ACOUSTICS

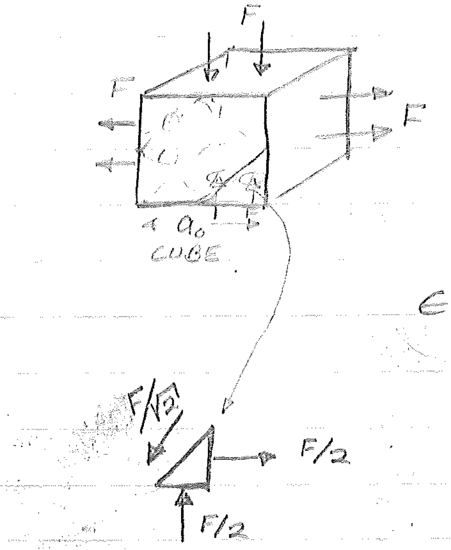
9-12-72

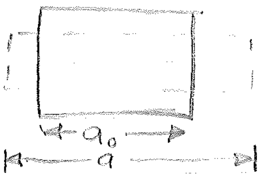
FOR NEXT MONDAY 1, 2, 4, 6, 8*, 9



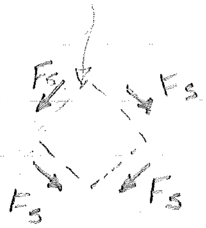
$\frac{F'}{A'} = \frac{F}{A}$

$\frac{S_{YY}}{\theta} = \frac{F'}{A'/\theta} = G =$ SHEAR MODULUS



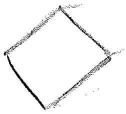


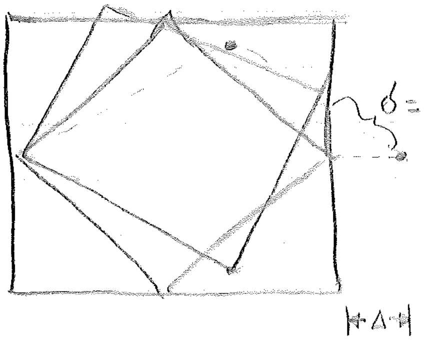
$\epsilon_{YY} = \frac{a - a_0}{a_0} = \frac{1}{Y}\frac{F}{a_0^2}(1 + \sigma)$



$\Rightarrow F_S = F/\sqrt{2}$

STRESS $= F/a_0^2$



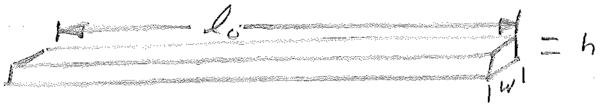


$\delta = \frac{1}{Y}\frac{F}{a_0^2}(1 + \sigma)\sqrt{2}$

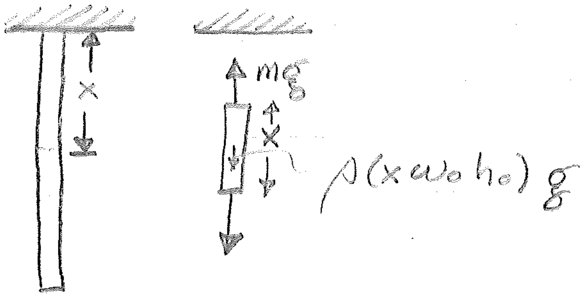
$1/Y \; F/a_0^2 (1 + \sigma)\sqrt{2} \big/ a_0/\sqrt{2}$

$\Rightarrow \theta$

$\frac{F/a_0^2}{\theta} = \frac{Y}{2(1 + \sigma)} = G$



m = MASS ; ρ = DENSITY

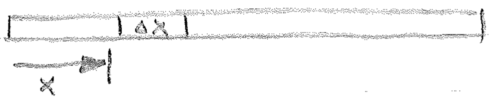


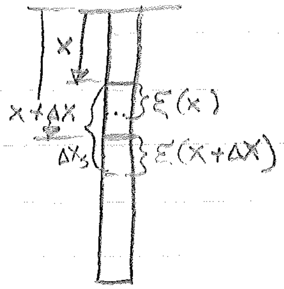
$$\Sigma F_x = 0 = -mg + \rho(x w_o h_o) g + F_x = 0$$

$$\Rightarrow F_x = mg - \rho(x w_o h_o) g$$

$$= \rho w_o h_o g [l_o - x]$$

$$S_{xx} = \rho g [l_o - x]$$





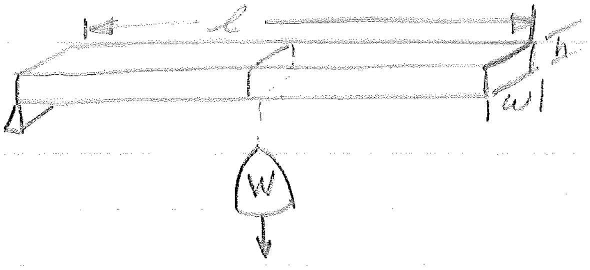
$$\epsilon_{xx} = \lim_{x \to 0} \frac{\Delta x_s - \Delta x}{\Delta x}$$

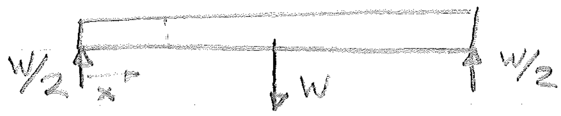
$$\Delta x_s = \Delta x + \mathcal{E}(x+\Delta x) - \mathcal{E}(x)$$

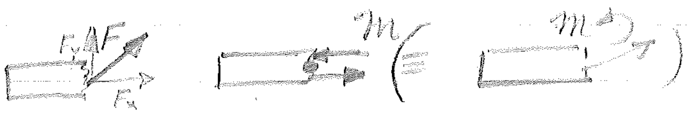
$$\Rightarrow \epsilon_{xx} = d\mathcal{E}/dx$$

$$= \frac{1}{Y} S_{xx}$$

$$\epsilon_{xx} = d\mathcal{E}/dx = \frac{1}{Y} [\rho g (l_o - x)]$$



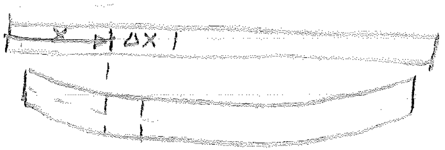


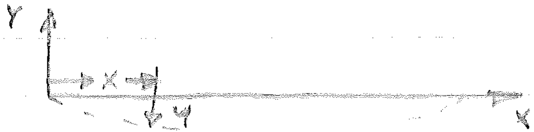


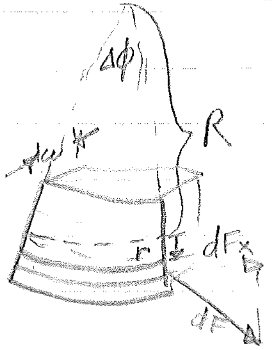
$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma \tau_z = 0$$

$$F_x = 0 \qquad \frac{W}{2} + F_y = 0 \qquad = -\frac{W}{2}x + m = 0$$

$$F_y = \left(-W/2\right) \qquad \Rightarrow m = \frac{W}{2}x$$







$$\frac{(R+r)\Delta\phi - R\Delta\phi}{R\Delta\phi} = \frac{r}{R} \quad \text{(STRAIN)}$$

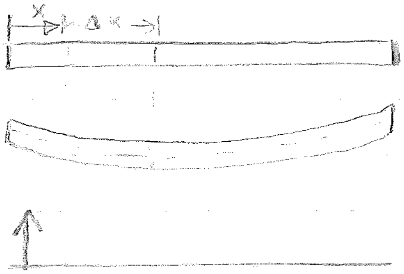
$$\Rightarrow \frac{r}{R} = \frac{1}{Y}\frac{dF_x}{w\,dr} \Rightarrow dF_x = \frac{Yrw\,dr}{R}$$

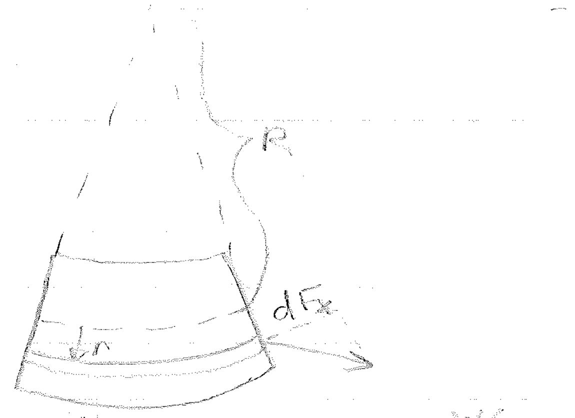
TAKE TORQE

$$2r\,dF_x = Yr^2w\,dr/R$$

$$\therefore m = \int_0^{h/2} 2Yr^2w\,dr/R$$

9-13-72





$\mathcal{M}$ = BENDING MOMENT = $\frac{W}{2}X$; $F_y = W/2$

$$\frac{r}{R} = \frac{1}{Yw} \, {dF_x}/{dr}$$

$$\Rightarrow dF_x = \frac{Yrw}{R} dr$$

$$2r \, dF_x = 2Yr^2 w \, dr/R$$

$$\mathcal{M} = \int_0^{h/2} \frac{2Yr^2 w}{R} dr$$

$$= \frac{2Yw}{3R} \left(\frac{h}{2}\right)^2 = Ywh^3/12R$$ ∋ R = RADIUS OF CURVATURE

FROM CALCULUS

$$R = \frac{\left[1 + \left(\frac{dY}{dx}\right)^2\right]^{3/2}}{dY^2/dx^2}$$

FOR SMALL $\frac{dY}{dx}$ (AS IN THIS CASE)

$$R = \left(\frac{d^2Y}{dx^2}\right)^{-1}$$

$$\Rightarrow \mathcal{M} = \frac{Ywh^3}{12} \frac{d^2Y}{dx^2} \quad \text{AND} \quad \mathcal{M} = \frac{W}{2}X$$

$$\therefore \quad \frac{W}{2}X = \frac{Ywh^3}{12} \frac{d^2Y}{dx^2}$$

$$\Rightarrow \quad \frac{dY^2}{dx^2} = \frac{6W}{Ywh^3}X$$

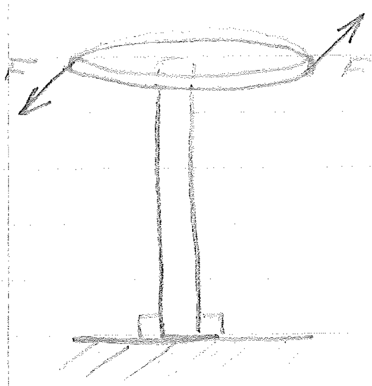
ERGO $\frac{dY}{dX} = \frac{3W}{Ywh^3}X^2 + C_1$

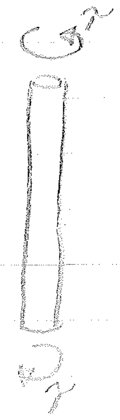
$$Y = \frac{W}{Ywh^3}X^2 + CX + C_2$$

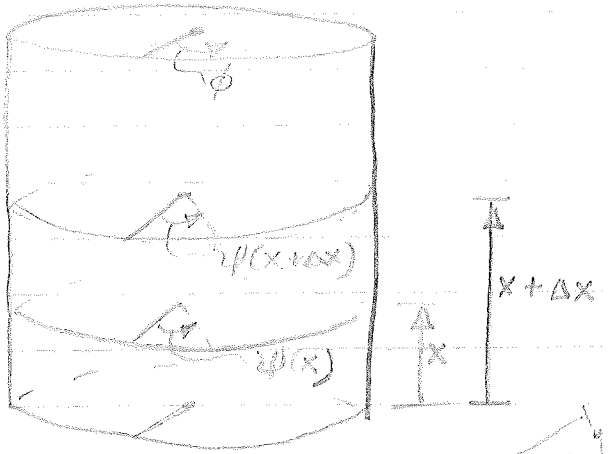
@ $x = \ell/2$, $\frac{dY}{dx} = 0$, @ $x = 0$, $Y = 0$

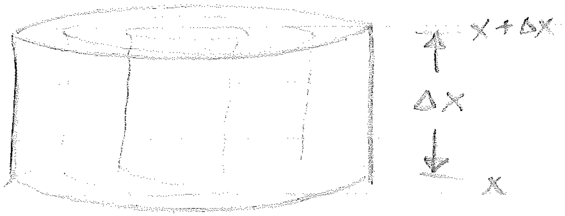
PLUG & CHUG:

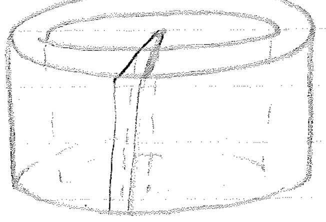
$$Y = \frac{W}{Ywh^3} X \left[X^2 - \frac{3}{4}L^2\right]$$

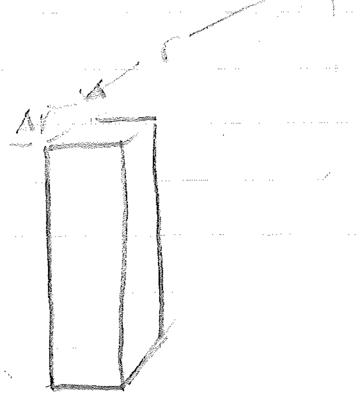




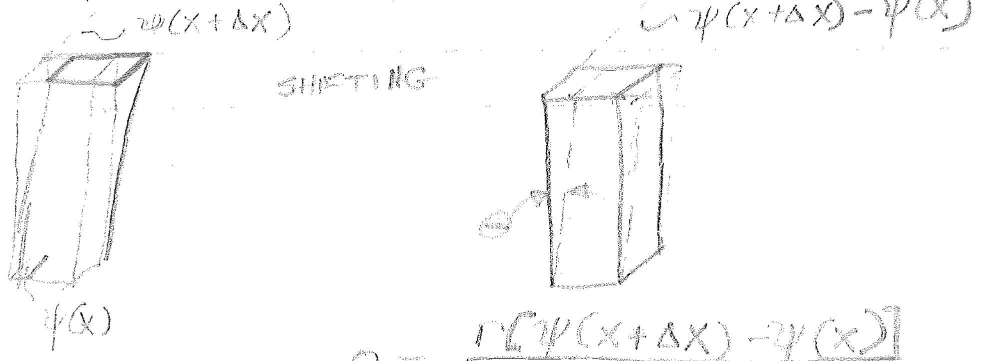






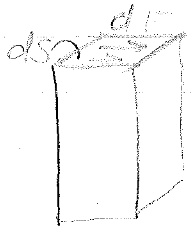


AFTER APLICATION OF τ



$$\theta = \frac{r[\psi(x+\Delta x) - \psi(x)]}{\Delta x}$$

$$\theta = r \, d\psi/dx$$



$$G = \frac{1}{\Theta} \, dF/ds$$

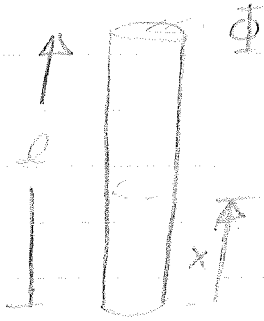
$$dF = G\Theta \, ds = G r \frac{d\psi}{dx} ds$$

$$d\gamma = r\,dF = G r^2 \frac{d\psi}{dx} ds$$

$$\Delta\gamma = G r^2 \frac{d\psi}{dx} 2\pi r \Delta R$$

$$> \gamma = \int_0^a G r^3 \, 2\pi \frac{d\psi}{dx} dr$$

$$= \frac{G\pi a^4}{2} \, d\psi/dx = \gamma_{EXTERNAL} = \text{CONSTANT}$$



γ

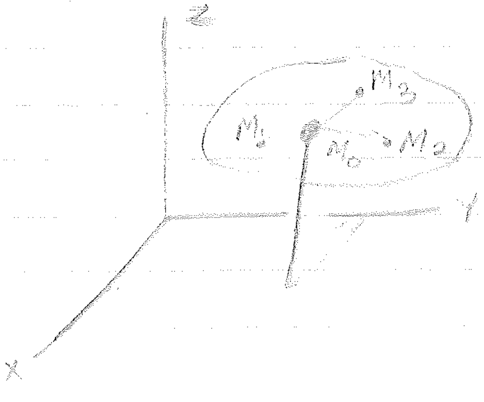
$$\Rightarrow \frac{d\psi}{dx} = C$$

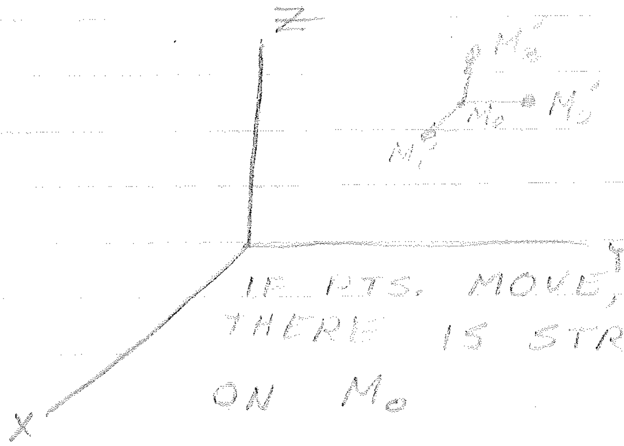
$$\psi = Cx + C_2 \nearrow 0$$

$$\psi = \frac{\Phi}{L} x$$

$$\gamma_{EXT} = \frac{G\pi a^4}{2} \frac{\Phi}{L}$$

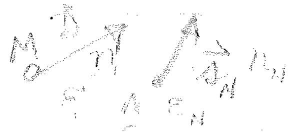
STRAIN @ A POINT





IF PTS. MOVE, THERE IS STRAIN ON M_0

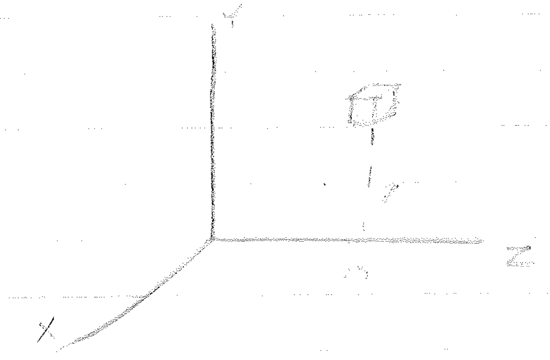
$$M(x, y, y) \qquad N(x+dx, y+dy, z+dz)$$



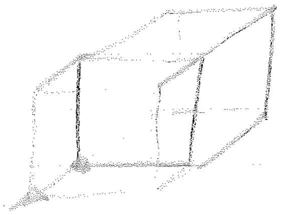
$$d\xi = \xi_N - \xi = (\delta\xi/\delta x)dx + (\delta\xi/\delta y)dy + \left(\frac{\delta\xi}{\delta z}\right)dz$$

$$d\eta = \eta_N - \eta \quad (\delta\eta/\delta x)dx + (\delta\eta/dy)dy + \left(\frac{\delta\eta}{\delta z}\right)dz$$

$$d\zeta \quad \zeta_N \quad \zeta = (\delta\zeta/dx)dx$$



LET $\frac{\delta E}{\delta X} \neq 0$, ALL OTHERS = 0



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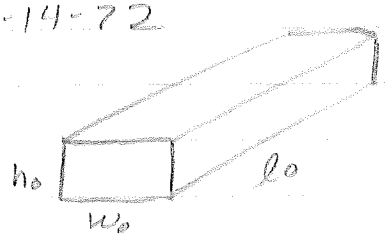


$\epsilon_{XX} = \frac{\ell - \ell_0}{\ell_0}$

$\epsilon_{YY} = \frac{W - W_0}{W_0}$

$\epsilon_{ZZ} = (h - h_0)/h_0$

$f(x+\Delta x) = f(x) + \frac{df}{dx}\Delta x$

$f(x+\Delta x, Y+\Delta Y)$

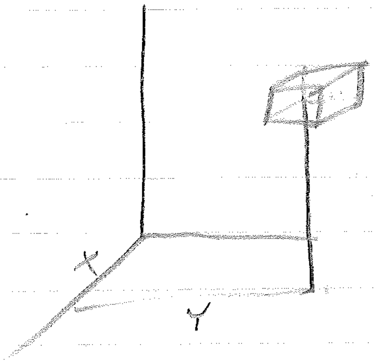
$= f(x,Y) + \frac{df}{\delta x}\Delta X$

$+ \frac{\delta f}{\delta Y}\Delta Y$

$f(x+\Delta x, Y+\Delta Y, Z+\Delta Z)$

$= f(x,YZ) + \frac{\delta f}{\delta x}\Delta X$

$+ \frac{\delta f}{\delta Y}\Delta Y + \frac{\delta f}{\delta Z}\Delta Z$



$\xi(x,Y,Z)$

$\eta(x,Y,Z)$

$\rho(x,Y,Z)$

$\xi(x+dx, Y+dY, z+dz) - \xi(x,Y,Z) = \frac{\delta\xi}{\delta x}dx + \frac{\delta\xi}{\delta Y}dY + \frac{\delta\xi}{\delta z}dz$

$\eta(x+dx, Y+dY, z+dZ) - \eta(x,Y,z) = \frac{\delta\eta}{\delta x}dx + \frac{\delta\eta}{\delta Y}dY + \frac{\delta\eta}{\delta z}dx$

$\rho(x+dx, Y+dY, z+dz) - \rho(x,Y,z) =$

FOR $\frac{\delta\xi}{\delta X} = 0.1$, ALL OTHERS = 0

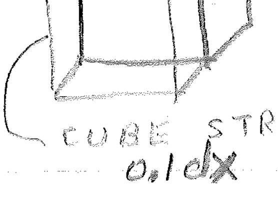
$H(x,Y,Z)$

$C(x+dx, Y+dY, Z)$

$\epsilon_{XX} = \delta\xi/\delta X$ ($= 0.1$

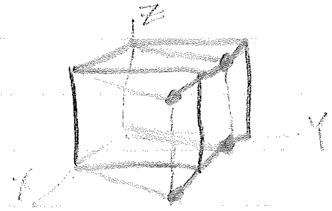
$\epsilon_{YY} = \delta\eta/dY$

$\epsilon_{ZZ} = \delta\rho/dZ$

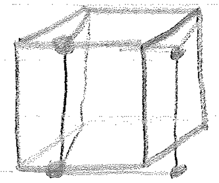


CUBE STRETCHED 0.1dX

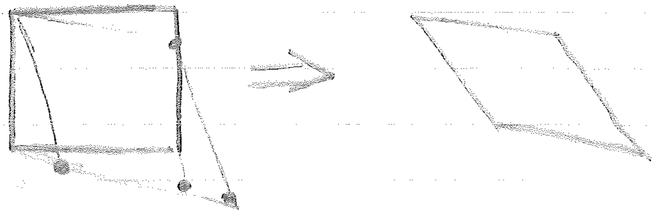
FOR $\frac{\delta \xi}{\delta Y} = 0.1$



FOR $\frac{\delta \eta}{\delta Z} = 0.1$



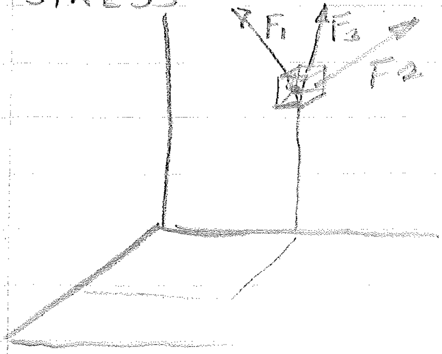
FOR $\frac{\delta \xi}{\delta Y} = \frac{\delta \eta}{\delta Z} = 0.1$



$\epsilon_{XX} = \frac{\delta \xi}{\delta Y}$ $\epsilon_{YY} = \frac{\delta \eta}{\delta Y}$ $\epsilon_{ZZ} = \frac{\delta \varphi}{\delta Z}$

$$\left.\begin{aligned} \epsilon_{XY} &= \frac{1}{2}\left(\frac{\delta \xi}{\delta Y} + \frac{\delta \eta}{\delta X}\right) = \epsilon_{YX} \\ \epsilon_{XZ} &= \frac{1}{2}\left(\frac{\delta \xi}{\delta Z} + \frac{\delta \varphi}{\delta X}\right) = \epsilon_{ZX} \\ \epsilon_{YZ} &= \frac{1}{2}\left(\frac{\delta \eta}{\delta Z} + \frac{\delta \varphi}{\delta Y}\right) = \epsilon_{ZY} \end{aligned}\right\} \text{SHEARING STRAINS}$$

STRESS



$F_1 \begin{cases} F_{XX} \\ F_{XY} \\ F_{XZ} \end{cases}$ $F_2 \begin{cases} F_{YX} \\ F_{YY} \\ F_{YZ} \end{cases}$ $F_3 \begin{cases} F_{ZX} \\ F_{ZY} \\ F_{ZY} \end{cases}$

$S_{XX} = F_{XX}/A$ | $S_{XY} = F_{XY}/A = S_{YX}$

$S_{YY} = F_{YY}/A$ | $S_{XZ} = F_{XZ}/A \quad S_{ZX}$

$S_{ZZ} = F_{ZZ}/A$ | $S_{Y3} = F_{13}/A = S_{ZY}$

STRESS AND STRAIN RELATIONS

$S_{XX} = C_{11}\epsilon_{XX} + C_{12}\epsilon_{YY} + C_{13}\epsilon_{ZZ} + C_{15}\epsilon_{XZ} + C_{16}\epsilon_{YZ}$

$S_{YY} = C_{21}\epsilon_{XX} + \ldots$

$\vdots$

$S_{YZ} = C_{61}\epsilon_{XX} + C_{62}\epsilon_{YY} + \ldots$

FOR HOMOGENEOUS ISOTROPIC SOLIDS:

$S_{xx} = (C_1 + C_2)\epsilon_{xx} + C_2 C_{yy} + C_3 \epsilon_{33}$

$S_{yy} =$ (ETC)

(CHECK IN TEXT)

$C_1 = \sigma Y / (1+\sigma)(1-2\sigma)$

$C_2 = Y / (1+\sigma)$

HARMONIC MOTION

$-KX = m\ddot{X}$

LET $\omega_0 = \sqrt{K/m}$

$\Rightarrow \ddot{x} + \omega_0^2 x = 0$

LET $X = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$

THEN $\ddot{X} = 0 + 0 + 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + \dots$

THUS $2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + \dots$

$= \omega_0^2 a_0 + \omega_0^2 a_1 t + \omega_0^2 a_2 t^2 + \omega_0^2 a_3 t^3 + \dots$

$(a_0\omega_0^2 + 2a_2) + [\omega_0^2 a_1 + 6a_3]t + (12a_4 + \omega_0^2 a_2)t^2$
$+ (\omega_0^2 a_3 + 20a_5)t^3 + \dots = 0$

$a_2 = \frac{-\omega_0^2}{2} a_0$; $a_3 = \frac{-\omega_0^2}{6} a_1$, $a_4 = \frac{-\omega_0^2}{12} a_2 = \frac{\omega^4}{2(12)} a_2$

$a_5 = \omega_0^4 / (20)(6)$

9·20·70
DUE TUES
2.3, 2.4, 2.7, 2.9, 2.11, 2.12, 2.14, 2.15, 2.16, 2.17

$m\ddot{x} + kx = 0$

$\ddot{x} + \omega_0^2 x = 0 \Rightarrow \omega_0 \sqrt{k/M}$

$x = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$

$\ddot{x} = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + \dots$

$\omega_0^2 x = \omega_0^2 a_0 + \omega_0^2 a_1 t + \omega_0^2 a_2 t^2 + \dots$

$\ddot{x} + \omega_0^2 x = (2a_2 + \omega_0^2 a_0) + (6a_3 + \omega_0^2 a_1)t + (12a_4 + \omega_0^2 a_2)t^2$
$+ (20a_5 + \omega_0 a_3)t^3 + \dots = 0$

$a_2 = -\omega_0^2/2\, a_0 \; ; \; a_3 = -\frac{\omega_0^2}{6} a_1$

$a_4 = -\frac{\omega_0}{12} a_2 = \frac{\omega_0^4}{24} a_0 \; ; \; a_5 = -\frac{\omega_0^2}{20} a_3 = \frac{\omega_0^4}{(6)(20)} a_1$

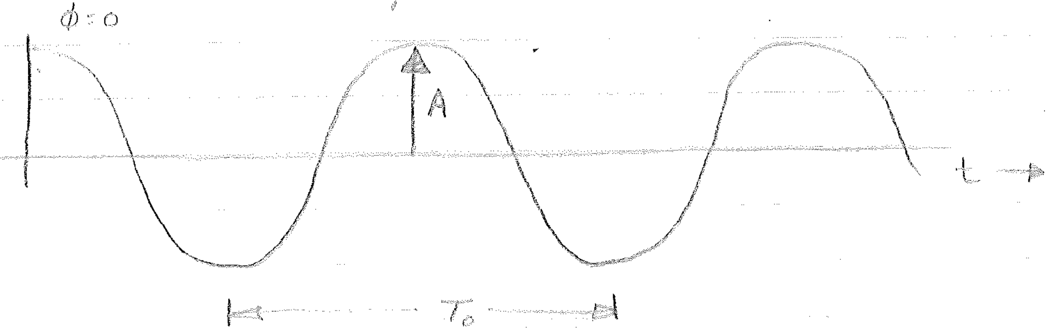
$\Rightarrow x = a_0 + a_1 t - \frac{\omega_0^2}{2} a_0 t^2 - \frac{\omega_0^2}{6} a_1 t^3 + \omega_0^4 a_1 t^3 + \dots$

$= a_0\left[1 - \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} + \dots\right] + \left[\frac{a_1}{\omega_0}(\omega_0 t - \frac{(\omega_0 t)^3}{3!} + \frac{(\omega_0 t)^5}{5!}\right.$

$= C \cos \omega_0 t + D \sin \omega_0 t$

$C = A \cos\phi \;\Big\}\; A = \sqrt{D^2 + C^2}$
$D = A \sin\phi \;\Big\}\; \tan\phi = D/C$

$\Rightarrow x = A\cos(\omega_0 t - \phi)$



$T_0 = 2\pi/\omega_0$
$f = 1/T$
ω_0 = ANGULAR FREQ

LET $X_1 = 3\cos\omega_0 t$ $\qquad \ddot{x}_1(t) + \omega_0^2 x_1(t) = 0$

$X_2 = 2\cos(\omega_0 t - \pi/2)$ $\qquad i\,\ddot{x}_2(t) + i\omega_0^2 x_2(t) = 0$

AND LET $x(t) = X_1(t) + i X_2(t)$

$\underset{\sim}{\ddot{x}}(t) = \ddot{x}_1(t) + i\ddot{X}_2(t)$ $\qquad i = \sqrt{-1}$

$\sim$ DENOTES COMPLEXITY

NOW WAVE EQUATION BECOMES:

$$\underset{\sim}{\ddot{X}} + \omega_0^2 \underset{\sim}{X} = 0$$

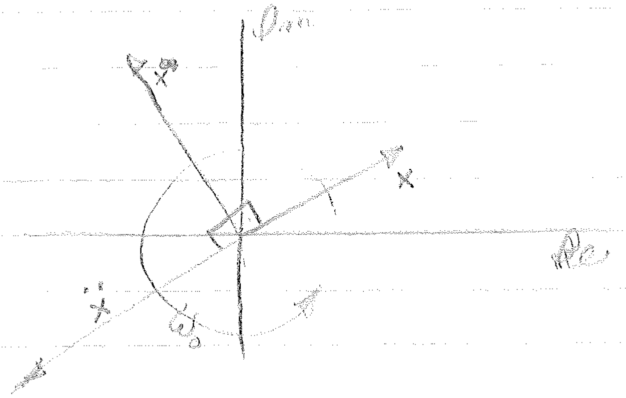
PHASE DIFFERENCE: $2\pi \frac{t_0 - t_1}{T_0}$

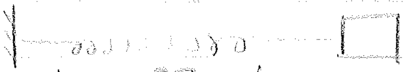
t_0 = TIME FIRST QUANITY IS MAX

t_1 = NEAREST TIME SECOND QUANTITY IS MAX

$\underset{\sim}{X} = \underset{\sim}{A} e^{i\omega t}$; $\underset{\sim}{\dot{X}} = i\omega \underset{\sim}{A} e^{i\omega t}$; $\ddot{X} = \omega^2 \underset{\sim}{A} e^{i\omega t}$

$= i\omega \underset{\sim}{X}$ $\qquad \ddot{X} = -\omega^2 X$





$$E = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} K x^2$$

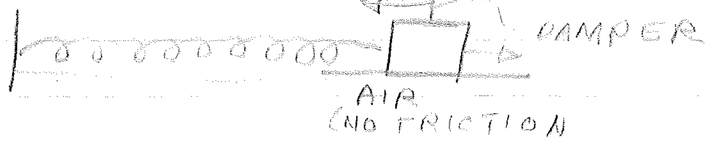
$$= \tfrac{1}{2} m A^2 \sin^2(\omega_0 t \ \phi) + \tfrac{1}{2} k A^2 \cos(\omega_0 t \ \phi)$$

$$\omega_0 = \sqrt{K/M}$$

$$E = \tfrac{1}{2} M A^2 \tfrac{K}{M} \sin^2(\omega_0 t - \phi) + \tfrac{1}{2} K A^2 \cos^2(\omega_0 t \ \phi)$$

$$= \underline{\tfrac{1}{2} K A^2} = \tfrac{1}{2} A \omega_0^2 M \qquad \text{(CONSTANT!)}$$

DAMPED HARMONIC OSCILLATORS

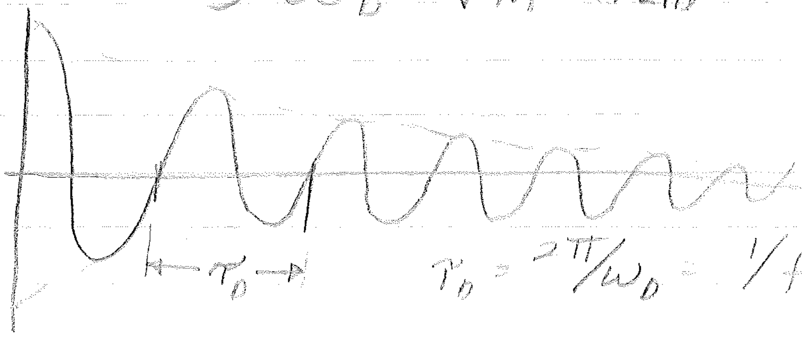


$$m\ddot{x} = Kx - R\dot{x} \qquad \text{LEX } \omega_0 = \sqrt{K/M} \qquad \alpha = \frac{R}{2M}$$

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0$$

$$\Rightarrow x = e^{-\alpha x} A \cos(\omega_D t - \phi)$$

$$\ni \omega_D = \sqrt{K/m - (R/2m)^2} = \sqrt{\omega_0^2 - \alpha^2}$$



$$\tau_D = 2\pi/\omega_D = 1/f_D$$

$$\frac{x_{n+2}}{x_n} = e^{\frac{2\pi\alpha}{\omega_n}}$$

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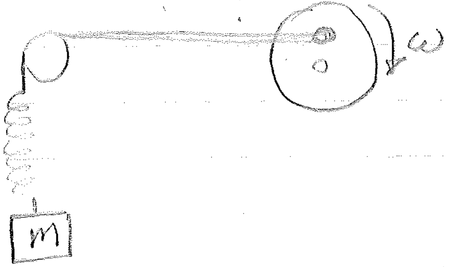
$$m\ddot{x} = -kx \Rightarrow x = A\cos(\omega_0 t - \phi) \Rightarrow \omega_0 = \sqrt{K/M} \quad T = \frac{2\pi}{\omega}$$

$$m\ddot{x} = -kx - R\dot{x} \Rightarrow x = e^{-\alpha t}[A\cos(\omega_D t - \phi)]$$

$$\omega_D = \sqrt{\omega_0^2 \alpha^2} \qquad \alpha = R/2m$$

NOW CONSIDER;

$$m\ddot{x} = kx - R\dot{x} + F\cos\omega t$$

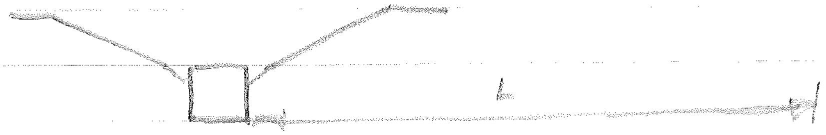


$$\Rightarrow m\ddot{x} + R\dot{x} + kx = F_0\cos\omega t$$

$$\therefore x = C\sin(\omega t - \theta)$$

$\dot{x} =$ (TO NOTES)

$$Y_n(x,t) = \sin\frac{n\pi}{L}x\left[A_n\cos\frac{n\pi c}{L}t + B_n\cos\frac{n\pi c}{L}t\right]$$
$$= \frac{A_n}{2}\left[\cos\frac{n\pi}{L}(x-ct) - \cos\frac{n\pi}{L}(x+ct)\right.$$
$$+ \frac{B_n}{2}\left[\sin\frac{n\pi}{L}(x+ct) + \sin\frac{n\pi}{L}(x-ct)\right]$$



$Y_0 = a\sin\omega t$

$$Y(x,t) = \begin{cases} 0 & X < {}^{x}/_{c} \\ a\sin\omega\left(t-\frac{x}{c}\right) & \frac{x}{c} < t < \frac{L}{c} \end{cases}$$

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PROGRESSIVE WAVE

$Y_1 = a\sin\left(\omega\left(t-\frac{x_1}{c}\right)\right)$

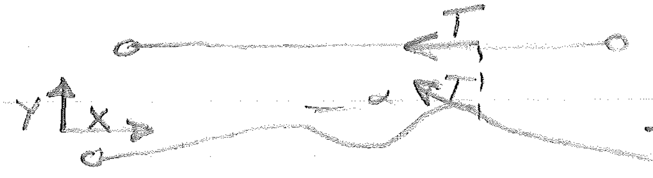
$Y_2 = a\sin\omega\left(t-\frac{x_2}{c}\right)$

PHASE DIFFERENCE $= \frac{\omega}{c}(x_2 - x_1)$

IF THE PHASE DIFFERENCE TWIXT THE MOTION @ TWO POINTS IS EXACTLY 2π, THEN WE SAY THE POINTS ARE APART BY λ

$C = \lambda f$

$\frac{\omega}{c} = k = \frac{2\pi}{\lambda}$



$F_Y = T\sin\alpha = T\tan\alpha \equiv -T\frac{\delta Y}{\delta X}$

WORK $= \vec{F}\cdot\vec{V} = F_y V_y = \left(-T\frac{\delta Y}{\delta X}\right)\frac{\delta Y}{\delta t}$

$dW = -T\frac{dY}{\delta X}\frac{\delta Y}{\delta t}dt$

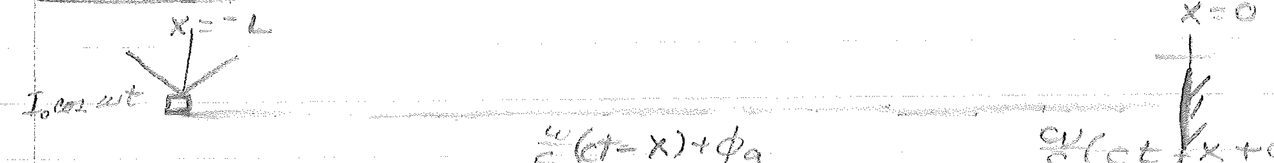
$$W_{AVE} = S = \frac{1}{\tau}\int_0^{\tau} -T\frac{\delta Y}{\delta X}\frac{\delta Y}{\delta t}dt$$

FOR A PROGRESSIVE WAVE

$$S = \frac{1}{\tau}\int_0^\tau -T\left[-a\frac{\omega}{c}\cos\omega\left(t-\frac{x}{c}\right)a\omega\cos\omega\left(t-\frac{x}{c}\right)\right]dt$$

$$= \frac{Ta^2\omega^2}{c\tau}\int_0^\tau \cos^2\omega\left(t-\frac{x}{c}\right)dt$$

$$= \frac{Ta^2\omega^2}{2c}$$



$$\overset{\frac{\omega}{c}(ct-x)+\phi_a}{} \qquad \overset{\frac{\omega}{c}(ct+x+\phi_b)}{}$$

$$Y(x,t) = a\cos(\omega t - kx + \phi_a) + b\cos(\omega t + kx + \phi_b)$$

$$a = |a|e^{i\phi_a} \qquad b = |b|e^{i\phi_b}$$

$$Y(x,t) = a e^{i(\omega t - kx)} + b e^{i(\omega t + kx)}$$

$$Z_S = \frac{\text{Y COMP. OF FORCE FROM LEFT HAND PORTION ON THE RIGHT HAND PORTION}}{\text{Y COMP. OF STRING VELOCITY}}$$

IMPEDANCE @ A POINT

$$Z_s = \frac{-T\,\delta Y/\delta x}{\delta Y/\delta t}$$

BOUNDRY CONDITIONS

$$Y(0,t) = 0 = a e^{i\omega t} + b e^{i\omega t} \Rightarrow a = -b$$

$$\Rightarrow Y = a e^{i\omega t}\left[e^{-ikx} - e^{ikx}\right]$$

$$= -2i\tilde{a}e^{i\omega t}\sin kx$$

$$= A\sin kx\, e^{i\omega t} \Rightarrow A = -2i\tilde{a}$$

$$\Rightarrow \frac{\delta Y}{\delta t} = +i\omega A \sin kx\, e^{i\omega t}$$

$$\therefore Z_s = \frac{-T\,\delta Y/\delta x}{\delta Y/\delta t} = \frac{-TkA\cos kx\, e^{i\omega t}}{+i\omega A\sin kx\, e^{i\omega t}}$$

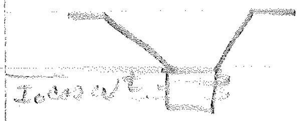
$$= -\frac{kT}{i\omega}\cot kx$$

$$= i\rho c \cot kx$$

4.

$$Z_s\Big|_{x=-L} = -i\rho c \cot kL$$

DRIVING POINT INPEDENCE IS Y COMP OF FORCE EXERTED BY THE DRIVER ON THE STRING OVER THE STRING'S VELOCITY = $Z_s\Big|_{x=-L}$

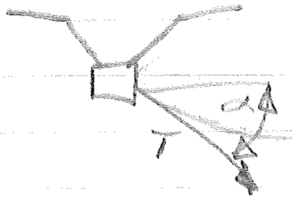


$$m\ddot{Y}_0 + R\dot{Y}_0 + KY_0 = BI_0 l\cos\omega t$$

$$mY_0 + R\dot{Y}_0 + KY_0 = BI_0 l e^{i\omega t}$$

$$\dot{Y}_0 = BI_0 l e^{i\omega t}/Z_m \Rightarrow Z_m = R + i\left(\omega m - \frac{K}{\omega}\right)$$

$$Y(x,t) = A \sin kx\, e^{i\omega t}$$



$$\Rightarrow m\ddot{Y}_0 + R\dot{Y}_0 + KY_0 = BIl e^{i\omega t} + T\frac{\delta Y}{\delta X}\Big|_{x=-L}$$

$$= (BI_0 l + TAk\cos kx)\Big|_{x=-L} e^{i\omega t}$$

$$= \underbrace{(BI_0 l + TAk\cos kL)}_{F_0} e^{i\omega t}$$

$$\Rightarrow m\ddot{Y}_0 + R\dot{Y} + KY_0 = F_0 e^{i\omega t}$$

$$Y_0 = \frac{F_0 e^{i\omega t}}{Z_m} = \frac{BI_0 l e^{i\omega t} + T\frac{\delta Y}{\delta X}\Big|_{x=-L}}{Z_m}$$

$$Z_m = \frac{BI_0 l e^{i\omega t}}{\dot{Y}_0} - \frac{T\frac{\delta Y}{\delta X}\Big|_{x=-L}}{\dot{Y}_0}$$

$$= \frac{BI_0 l e^{i\omega t}}{\dot{Y}_0} - \frac{-T\,\delta Y/\delta X}{\delta Y/\delta t}\Big|_{x=-L}$$

$$Z_{DP} + Z_m = \frac{BI_0 l e^{i\omega t}}{\dot{Y}_0} \Rightarrow \dot{Y}_0 = \frac{BI_0 l e^{i\omega t}}{Z_m + Z_{DP}}$$

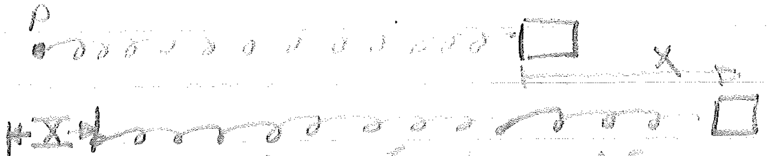
BEFORE ATTATCHING STRING

$$\dot{Y}_0 = \frac{BI_0 l e^{i\omega t}}{Z_m}$$

$$T\frac{dY}{dt}$$

10-2-72

TEST OVER ELASTICITY, HARMONIC MOTION, WAVES ON STRINGS ON THURS



$-k(x-\bar{X}) - R\dot{X} = m\ddot{X}$

$Y(x,t) = \underline{a}\, e^{i(\omega t - kx)} + \underline{b}\, e^{i(\omega t + kx)}$

$\underline{a} = \underline{b}$ (FROM BOUNDRY)

$\Rightarrow Y(x,t) = \underline{A} \sin kt\, e^{i\omega t} \Rightarrow \underline{A} = -2\underline{a}i$

SUMMING FORCES

$m\ddot{Y}_D + R\dot{Y}_D + KY_D = BI_0 l e^{i\omega t} + T \frac{\delta Y}{\delta x}\Big|_{x=-L}$

$$\dot{Y}_D = \frac{BI_0 l e^{i\omega t} + T \frac{\delta Y}{\delta x}\Big|_{x=-L}}{Z_m}$$

$$\Rightarrow Z_m = \frac{BI_0 l e^{i\omega t}}{\dot{Y}_D} - \frac{-T \frac{\delta Y}{\delta x}\Big|_{x=-L}}{\delta Y/\delta t\Big|_{x=-L}}$$

$$Z_{DP} + Z_m = \frac{BI_0 l e^{i\omega t}}{\dot{Y}_D}$$

$$\Rightarrow \dot{Y}_D = \frac{BI_0 l e^{i\omega t}}{Z_m + Z_{DP}} \quad ; Z_{DP} = -i\rho c \cot kL$$

$$\Rightarrow Z_m + Z_{DP} = R + i\left(\omega m - \frac{K}{\omega}\right) - i\rho c \cot\frac{\omega}{c}L$$

$= \infty$ WHEN $\frac{\omega}{c}L = n\pi$

$Y(x,t) = \underset{\sim}{A} \sin kx\, e^{i\omega t} \xrightarrow{Real} A \sin kx \cos(\omega t + \phi)$

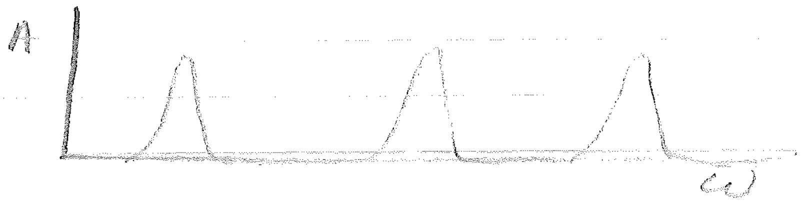
VELOCITY OF VOICE COIL AND STRING ARE THE SAME:

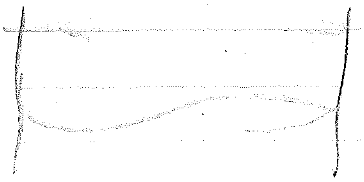
$$\frac{B E_0 l\, e^{i\omega t}}{Z_m + Z_{OD}} = \frac{\delta Y}{\delta t}\Big|_{x=L} = A i \omega \sin kL\, e^{i\omega}$$

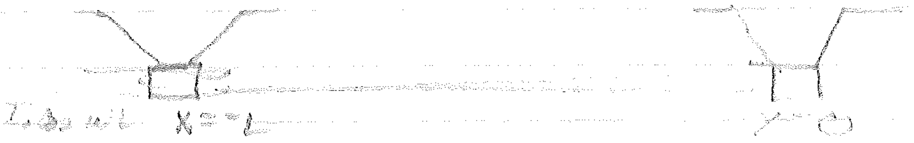
$$\tilde{A} = \frac{B E_0 l_0}{i\omega \sin kL[R + i(\omega m - \frac{K}{L}) - i\rho c \cot kL}$$

$$|A| = \frac{B E_0 l}{\omega[\{(\omega m - \frac{K}{\omega})\sin kL - \rho c \cos kL\} - iR\sin kL}$$

IF $\frac{\omega}{c} L = n\pi$, A IS REALLY BIG







$$Y = a e^{i(\omega t - kx)} + b e^{i(\omega t + kx)}$$

$$Z_0 = R_0 + i(\omega M_0 - K_0/\omega); \quad Z_T = R_T + i(\omega M_T - \frac{k}{\omega}$$

$$M_T \ddot{Y}_T + R_T \dot{Y}_T + K Y_T = -T \frac{\partial Y}{\partial x}\Big|_{x=0}$$

$$= -Ta(ik)e^{i(\omega t - kx)} + bike^{i\omega t + ikx}\Big|_{x=0}$$

$$= ikT[a-b]e^{i\omega t}$$

$$\Rightarrow \dot{Y}_T = \frac{ikT[a-b]e^{i\omega t}}{Z_T}$$

$$\dot{Y}_T = \frac{\partial Y}{\partial t}\Big|_{x=0}$$

$$\Rightarrow \frac{ikT[a-b]\cancel{e^{i\omega t}}}{Z_T} = i\omega(a+b)\cancel{e^{i\omega t}}$$

$$\frac{a+b}{a-b} = \frac{ikT}{Z_T(i\omega)} = \frac{\rho c^2 \omega/c}{Z_T \omega} = \frac{\rho c}{Z_T}$$

$$Y = a e^{i(\omega t - kx)} + a \frac{\rho c - Z_T}{\rho c + Z_T}$$

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$$Z_T = R_T + i\left(\omega m_T - \frac{K}{\omega_T}\right)$$

$$Y(x,t) = a e^{i(\omega t - kx)} + b e^{i(\omega t + kx)}$$

$$= a e^{i\omega t}\left[e^{-ikx} + \frac{\rho c - Z_T}{\rho c + Z_T} e^{ikx}\right]$$

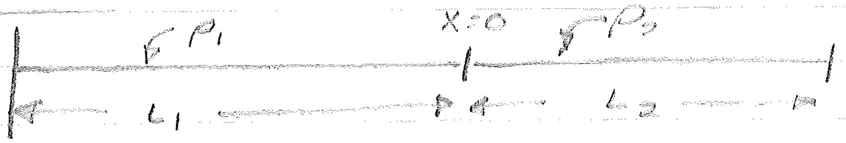
$$Z_s = \frac{-T\, \delta Y/\delta x}{\delta Y/\delta t}$$

$$= \frac{-Ta e^{i\omega t}\left[-ik e^{-ikx} + ik\frac{\rho c - Z_T}{\rho c + Z_T} e^{ikx}\right]}{i\omega a e^{i\omega t}\left[e^{-ikx} + \frac{\rho c - Z_T}{\rho c + Z_T} e^{ikx}\right]}$$

$$= \rho c \left[\frac{e^{-ikx} - \frac{\rho c - Z_T}{\rho c + Z_T} e^{ikx}}{e^{ikx} + \frac{\rho c - Z_T}{\rho c + Z_T} e^{-ikx}}\right]$$

$$= \rho c \frac{Z_T - i\rho c \tan kx}{\rho c - iZ_T \tan kx}$$

$$\lim_{Z_T \to \infty} Z_s = i\rho c \cot kX$$



$$Y_1 = f_1(x - c_1 t) + g_1(x + c_1 t) \qquad c_1 = \sqrt{T/\rho_2}$$

$$Y_2 = f_2(x - c_2 t) + g_2(x + c_2 t) \qquad c_2 = \sqrt{T/\rho_1}$$

$$Y_1(0,t) = Y_2(0,t)$$

$$\left.\frac{\delta Y_1}{\delta x}\right|_{x=0} = \left.\frac{\delta Y_2}{\delta x}\right|_{x=0}$$

$$Y_1(x,t) = a_1 e^{i(\omega t - k_1 x)} + b_1 e^{i(\omega t + k_1 x)}$$

$$Y_2(x,t) = a_2 e^{i(\omega t + k_2 x)} + b_2 e^{i(\omega t + k_2 x)}$$

$a_1 + b_1 = a_2$ FROM $Y_1(0,t) = Y_2(0,t)$

$-ik_1 a_1 + ik_1 b_1 = -ik_2 a_2$ $\quad \frac{\delta Y_1(0,t)}{\delta t}\Big|_t = \frac{\delta Y_2(0,t)}{\delta t}$

$\Rightarrow a_1 - b_1 = \frac{k_2}{k_1} a_2 = \frac{c_1}{c_2} a_2$

$\frac{c_2}{c_1} = \frac{a_1 + b_1}{a_1 - b_1} \Rightarrow \frac{b_1}{a_1} = \frac{\frac{c_2}{c_1} - 1}{\frac{c_2}{c_1} + 1} = \frac{c_2 - c_1}{c_2 + c_1}$

$Y_1 = a_1 e^{i(\omega t - kx)} + b_1 e^{i(\omega t + k_1 x)}$

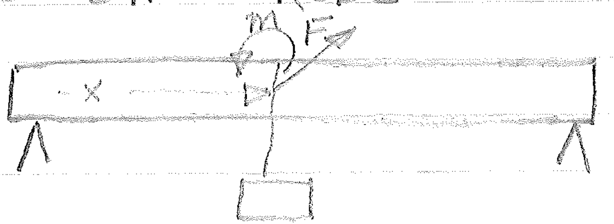
$Y_2(x,t) = a_2 e^{i(\omega t - k_2 x)}$

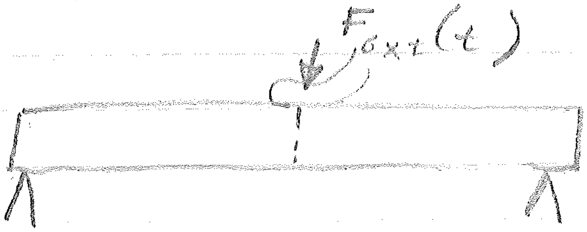
$Y_1\Big|_{x=0} = a_1 e^{i\omega t}$

$Y_2\Big|_{x=0} = b_1 e^{i\omega t} = \frac{c_2 - c_1}{c_2 + c_1} a_1 e^{i\omega t} =$

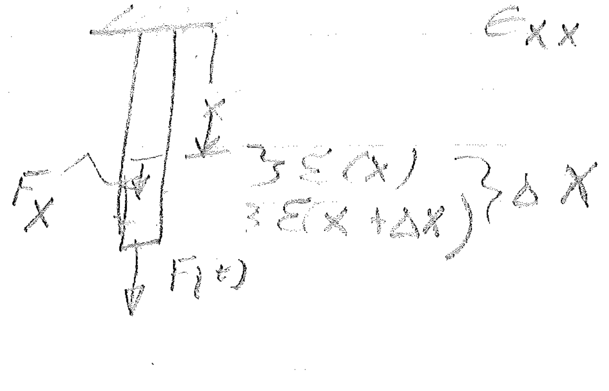
$= \left(\frac{c_2 - c_1}{c_2 + c_1}\right) Y_1\Big|_{x=0}$

WAVES ON RODS





SIMPLE CASE:



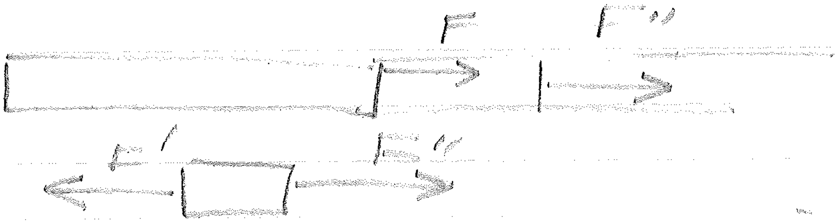
$\epsilon_{xx} = \lim_{\Delta x \to 0} \frac{\Delta x_\epsilon - \Delta x}{\Delta x}$

$\Rightarrow \epsilon_{xx} = \frac{\delta \xi}{\delta x}$

$\frac{\delta \xi}{\delta x} = \frac{1}{Y} S_{xx} = \frac{1}{Y} \frac{F_x}{A}$

$F_x = YA \frac{\delta \xi}{\delta x}$

STRESS-STRAIN
RELATIONSHIP



$$F'' + F = ma$$

$$F''_x + F_x = ma_x$$

$$F''_x - F_x = ma_x$$

$$F_x(x+\Delta x) - F_x(x) = \rho A \Delta x \frac{\delta^2 \xi}{\delta t^2}\Big|_{x+\Delta x/2}$$

$$\frac{\delta F_x}{\delta x} = \rho A \frac{\delta^2 \xi}{dt^2}$$

$$YA \frac{\delta^2 \xi}{\delta x^2} = \rho A \frac{\delta^2 \xi}{\delta t^2}$$

$$\frac{Y}{\rho} \frac{\delta^2 \xi}{\delta x^2} = \frac{\delta^2 \xi}{\delta t^2} \Rightarrow c^2 \frac{\delta^2 \xi}{\delta x^2} = \frac{\delta^2 \xi}{\delta t^2}$$

$$\xi(x,t) = X(x)H(t)$$
$$= \left[C \cos\left(\frac{\omega}{c} x\right) + A \sin \frac{\omega}{c} x\right] \cos \omega t$$
$$+ \left(D \cos \frac{\omega}{c} x + B \sin \frac{\omega}{c} x\right] \sin \omega t$$

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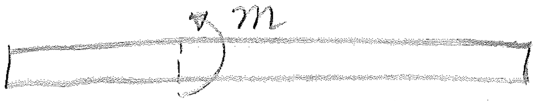
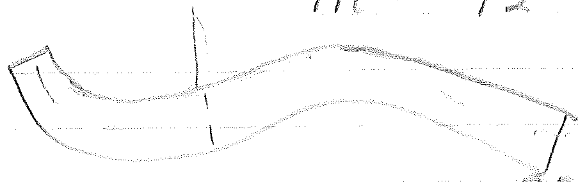
TRANSVERSE WAVES IN RODS

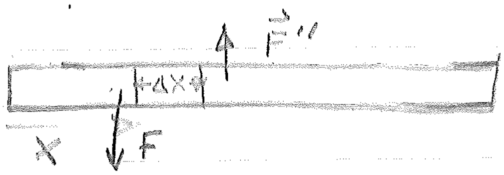


$$\mathcal{M} = \frac{Y\omega h^3}{12}\frac{d^2y}{dx^2}$$



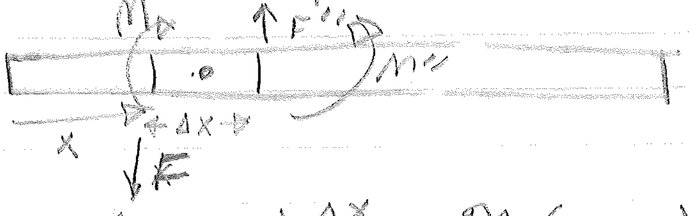
$Y(x,t)$

$$\mathcal{M} = \frac{Y\omega h^3}{12}\frac{\delta^2 Y}{\delta X^2}$$



$$F_Y(x+\Delta x) - F_Y(x) = \rho\omega h\Delta x\left.\frac{\delta^2 Y}{\delta t^2}\right|_{x+\frac{\Delta x}{2}}$$

$$\Rightarrow \frac{\delta F_Y}{\delta X} = \rho\omega h\frac{\delta^2 Y}{\delta t^2}$$



$$F_Y(x+\Delta x)\frac{\Delta x}{2} + \mathcal{M}(x+\Delta x) - \mathcal{M}(x) + F_Y(x)\frac{\Delta x}{2}$$
$$= I\alpha_z$$
$$= \frac{1}{12}\rho\omega h\Delta x\ \Delta x^2\alpha_z$$

$$\frac{F_Y(x+\Delta x) + F_Y(x)}{2} + \frac{\mathcal{M}(x+\Delta x) - \mathcal{M}(x)}{\Delta x} = \frac{1}{2}\rho\omega L(\Delta x)^2\alpha_z$$

$$\Rightarrow F_Y(x) + \frac{\delta\mathcal{M}}{\delta X} = 0 \Rightarrow F_Y = -\frac{\delta\mathcal{M}}{\delta X}$$

$$\frac{\delta F_y}{\delta x} = \rho w h \frac{\delta^2 y}{\delta t^2} \quad ; \quad F_y = -\frac{\delta M}{\delta x} ; M = \frac{Y w h^3}{12}\frac{\delta^2 y}{\delta x^2}$$

$$\Rightarrow -\frac{\delta^2 M}{\delta x^2} = \rho w h \frac{\delta^2 y}{\delta t^2}$$

$$-\frac{Y w h^3}{12}\frac{\delta^4 y}{\delta x^4} = \rho w h \frac{\delta^2 y}{\delta t^2}$$

$$c^2 I^2 \frac{\delta^4 y}{\delta x^4} = -\frac{\delta^2 y}{\delta t^2} \qquad c = \sqrt{Y/\rho} \quad I = \frac{h}{\sqrt{12}}$$

$$M = \frac{Y w h^3}{12}\frac{\delta^2 y}{\delta x^2} \; ; \; F_y = -\frac{\delta M}{\delta x} \; ; \; c^2 I^2 \frac{\delta^4 y}{\delta x^4} = \frac{\delta^2 y}{\delta t^2}$$

$$Y(x,t) = X(x)\, H(t)$$

$$c^2 I^2 H(t) \frac{d^4 X}{d x^2} = -X \frac{d^2 H}{d t^2}$$

$$-\frac{c^2 I^2}{X}\frac{d^4 X}{d x^4} = \frac{1}{H}\frac{d^2 H}{d t^2} = -\omega^2$$

$$\frac{d^2 H}{d t^2} = -\omega^2 H \Rightarrow H = b_1 \cos\omega t + b_2 \sin\omega t$$

$$\boxed{\frac{d^4 X}{d x^4}} = \frac{\omega^2}{c^2 I^2} X = \alpha^4 X \quad \Rightarrow \quad \alpha = \sqrt{\frac{\omega}{c I}}$$

$$X = a_1 \cos\alpha x + b_1 \sin\alpha x + c_1 \cosh\alpha x + d_1 \sinh\alpha x$$

$$\Rightarrow Y(x,t) = \left[A_1 \cos\alpha x + A_2 \sin\alpha x + A_3 \cosh\alpha x + A_4 \sinh\alpha x\right] \cos\omega t + \left[B_1 \cos\alpha x + B_2 \sin\alpha x + B_3 \cosh\alpha x + B_4 \sinh\alpha x\right] \sin\omega t$$

IF ALL BUT $A_1 = 0$

$$Y(x,t) = A_1 \cos \alpha x \cos \omega t$$
$$= A_1 [\cos(\alpha x + \omega t) + \cos(\alpha x - \omega t)]$$
$$= \frac{A_1}{2} [\cos \alpha [x + vt] + \cos \alpha [x - vt)] \Rightarrow V = \frac{\omega}{\alpha}$$

$$V = \frac{\omega}{\alpha} = \frac{\omega}{\sqrt{\frac{\omega}{cI}}} = \sqrt{\omega c I}$$

BOUNDRY CONDITIONS

$$Y(0,t) = Y(L,t) = 0$$
$$\frac{\delta Y}{\delta x}\Big|_{0,t} = 0 = \frac{\delta Y}{\delta x}\Big|_{Lt}$$

$$Y(0,t) = [A_1 + A_3] \cos \omega t + [B_1 + B_3] \sin \omega t = 0$$
$$A_3 = -A \quad ; \quad B_3 = -B_1$$
$$\Rightarrow Y(x,t) = [A_1 (\cos \alpha x - \cosh \alpha x) + A_2 \sin \alpha x + A_4 \sinh \alpha x] \cos \omega t$$
$$+ [B_1 (\qquad) + B_2 \qquad + B_4 \qquad] \sin \omega t$$

$$\frac{dY}{dx}\Big|_{0,t} = 0 = (A_2 + A_4) \cos \omega t + [B_2 + B_4) \sin \omega t$$
$$\Rightarrow A_2 = -A_4 \quad ; \quad B_2 = -B_4$$

$$\Rightarrow Y(x,t) = A_1 (\cos \alpha x - \cosh \alpha x) + A_2 (\sin \alpha x - \sinh \alpha x)] \cos \omega t$$
$$+ [B_1 (\qquad) + B_2 (\qquad)] \sin \omega t$$

FOR $Y(L,t) = 0$

$$A_1 (\cos \alpha L - \cosh \alpha L) + A_2 (\sin \alpha L - \sin \alpha L) = 0$$

FOR $\frac{dY}{dX}\Big|_{x=L}$

$$A_1 (-\sin \alpha L - \sinh \alpha L) + A_2 (\cos \alpha L - \cosh \alpha L) = 0$$

SOLVING FOR $\frac{A_2}{A_1}$;

$$\frac{A_2}{A_1} = -\frac{\cos\alpha L - \cosh\alpha L}{\sin\alpha L - \sinh\alpha L} = \frac{\sin\alpha L + \sinh\alpha L}{\cos\alpha L - \cosh\alpha L}$$

$$= -\left[\cos^2\alpha L - 2\cos\alpha L\cosh\alpha L + \cosh^2\alpha L\right] = \sin^2\alpha L - \sinh^2\alpha L$$

$$\Rightarrow 2\cos\alpha L\cosh\alpha L = \sin^2\alpha L + \cos^2\alpha L - \left[\sinh^2\alpha L - \cosh^2\alpha L\right]$$

$$\Rightarrow 2\cos\alpha L\cosh\alpha L = 2$$

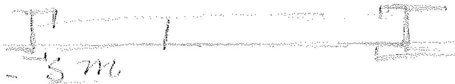
$$\therefore \cos\alpha L\cosh\alpha L = \underline{1} \qquad \ni \alpha = \sqrt{\frac{\omega}{CI}}$$

$$\alpha L = \frac{3.01}{2}\pi,\ \frac{5}{2}\pi,\ \frac{7}{2}\pi,\ \ldots = \sqrt{\frac{\omega}{CI}}\,L$$

$$\Rightarrow \omega = \left(\frac{3.01\pi}{2L}\right)^2 CI,\ \left(\frac{5\pi}{2L}\right)^2 CI,\ \ldots$$

EIGEN FREQ

TUES.



$F_y = -\frac{\delta m}{\delta x}$

$m = \frac{Ywh^3}{12} \frac{\delta^2 Y}{\delta x^2}$

$C^2 I^2 \frac{\delta^4 Y}{\delta x^4} = -\frac{\delta^2 Y}{\delta t^2}$

$$Y(x,t) = (A_1 \cos \alpha t + A_2 \sin \alpha t + A_3 \cosh \alpha x + A_4 \sinh \alpha x) \cos \omega t$$
$$+ (B_1 \cos \alpha x + B_2 \sin \alpha x + B_3 \cos \alpha x + B_4 \sinh \alpha x) \sin \omega t$$

BOUNDRY CONDITIONS:

$Y(0,t) = 0 \Rightarrow A_3 = -A_1$

$\frac{\delta Y}{\delta x}\Big|_{0,t} = 0 \Rightarrow A_4 = -A_2$

$Y(L,t) = 0 \Rightarrow \frac{A_2}{A_1} = -\frac{\cos \alpha L - \cosh \alpha L}{\sin \alpha L - \sinh \alpha L} = \frac{B_2}{B_1}$

$\frac{\delta Y}{\delta t}\Big|_{L,t} = 0 \Rightarrow \frac{A_2}{A_1} = \frac{\sin \alpha L + \sinh \alpha L}{\cos \alpha L - \cosh \alpha L} = \frac{B_2}{B_1}$

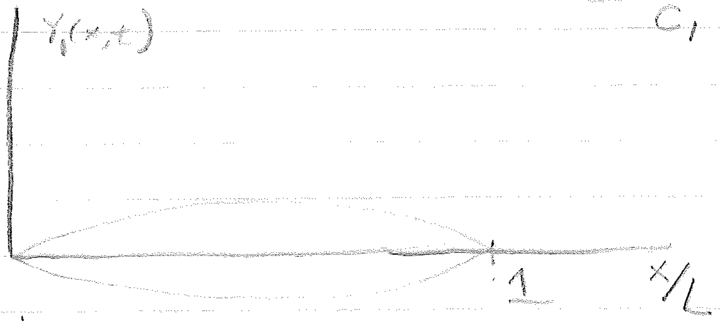
$\Rightarrow \cosh \alpha L \cos \alpha L = 1$

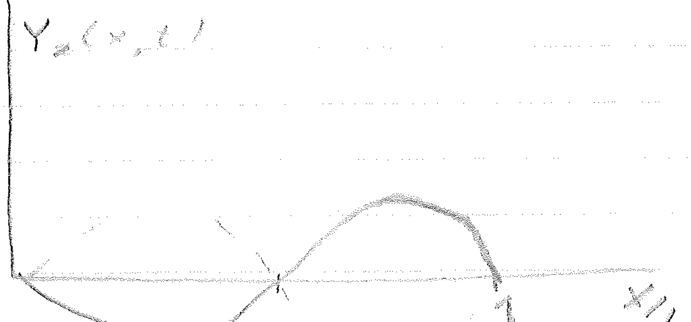
$\Rightarrow \alpha L = \frac{3.01\pi}{2}, \frac{5}{2}\pi, \frac{7}{2}\pi, \ldots \qquad \Rightarrow \alpha = \sqrt{\frac{\omega}{CI}}$

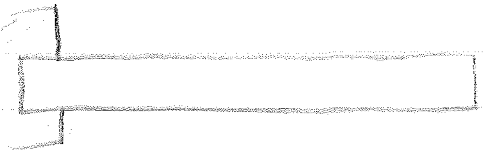
$\Rightarrow \omega = \left(\frac{3.01}{2\pi}\right)^2 CI, \left(\frac{5\pi}{2L}\right)^2 CL, \ldots$

$f = 3.56 \frac{CI}{L^2}, 9.8 \frac{CI}{L^2}$

$$Y_1(x,t) = \left[\left(\cos \frac{3.01\pi}{2L} x - \cosh \frac{3.01}{2L} x\right) - .983 \left(\sin \frac{3.01\pi}{2L} x - \sinh \frac{3.01\pi}{2L} x\right)\right] \underbrace{\left[A_1 \cos \omega_1 t + B_1 \sin \omega_1 t\right]}_{C_1 \cos(\omega_1 t + \phi_1)}$$







$Y(x,t) = (A_1 \cos \alpha x + A_2 \sin \alpha x + A_3 \cosh \alpha x + A_4 \sinh(\alpha x)) \cos \omega t$
$\quad + (B_1 \quad B_2 \quad B_3 \quad B_4) \sin \omega t$

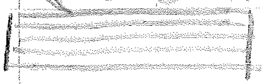
B. CONDITIONS: $Y(0,t) = 0 \Rightarrow A_3 = -A_1$; $\frac{\delta Y}{\delta x}\Big|_{0,t} \Rightarrow A_4 = -A_2$

@$_{x=L}$, $M = 0 \Rightarrow \frac{\delta^2 Y}{\delta x^2}\Big|_{x=L} = 0$

$F_Y = 0 \Rightarrow \frac{\delta^3 Y}{\delta x_2} = 0$ @ $x = L$

(DUE TUES.
WORK OUT FREE ROD OUT, FOR 3RD EIGEN FREQ

(NUMBERS IN BOOK)

3-6 ⇔ WORK

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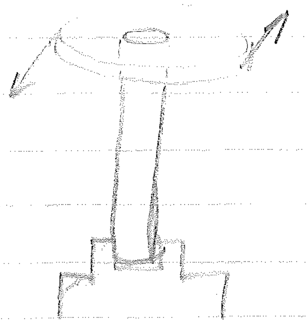
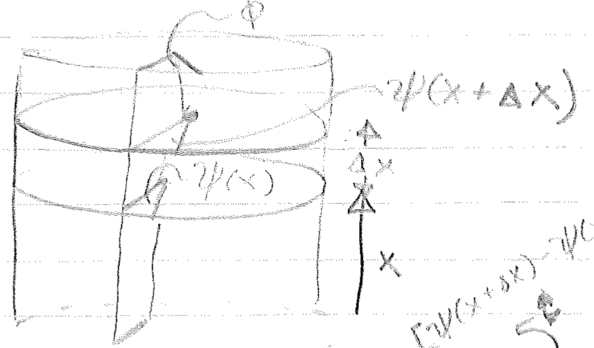
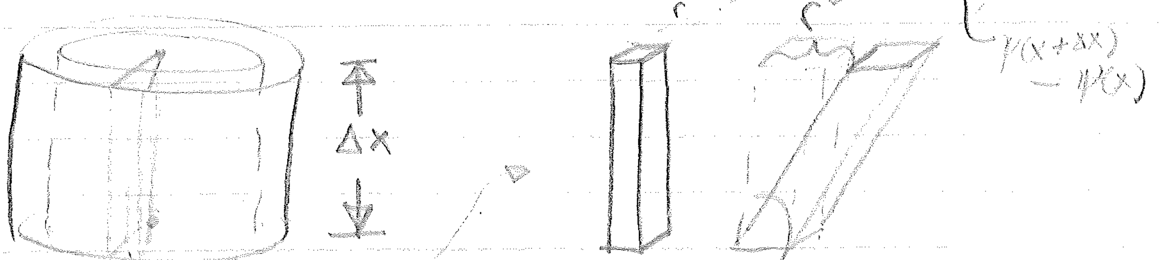
TORSIONAL WAVES

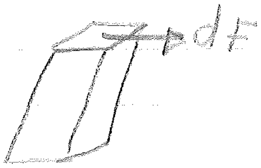






$\theta = \frac{r[\psi(x+\Delta x) - \psi(x)]}{\Delta x}$

$\Rightarrow \theta = r\frac{d\psi}{dx}$



$\frac{dF}{dA} / \theta = G$

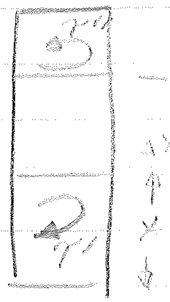
$\Rightarrow dF = G\theta dA = G r \frac{\delta\psi}{dx} dA$

$r\,dF = G r \frac{d\psi}{dx} dA\, r$ $\qquad \Delta\tau = G \frac{d\psi}{dr}(2\pi r dr) r$

$\tau = \int_0^a G r (\frac{\delta\psi}{\delta x}\, 2\pi r) dr = \tau_{EXT}$

$= G \frac{d\psi}{\delta x} \frac{\pi a^4}{2}$

$$\tau(x,t) = G\frac{\delta\psi}{\delta x}\frac{\pi a^4}{2}$$



$$\tau(x+\delta x,t)-\tau(x,t) = \overset{\frac{1}{2}I\alpha}{\left[\frac{1}{2}\rho A_{,}x\,\pi a^2\right]a^2} \quad \delta^2.$$

$$\Rightarrow \frac{\delta\tau}{\delta x} = \frac{\pi a^4}{2}\rho\frac{\delta^2\psi}{\delta t^2}$$

$$\Rightarrow G\frac{\delta^2\psi}{\delta x^2}\pi\frac{a^4}{2} = \pi\frac{a^4}{2}\rho\frac{\delta^2\psi}{\delta t^2}$$

$$\therefore \frac{G}{\rho}\frac{\delta^2\psi}{\delta x^2} = \boxed{\frac{\delta^2\psi}{\delta t^2} = c^2\frac{\delta^2\psi}{\delta x^2}} \Rightarrow c = \sqrt{\frac{G}{\rho}}$$

$$\Rightarrow \psi(x,ct) = f_1(x-ct) + f_2(x+ct)$$

$$\psi(x,t) = X(x)H(t)$$

$$\frac{c^2}{X}\frac{d^2X}{dx^2} = \frac{1}{H}\frac{d^2H}{dt^2} = -\omega^2$$

$$H(t) = b_1\cos\omega t + b_2\sin\omega t$$

$$X(x) = a_1\cos\frac{\omega}{c}x + a_2\sin\frac{\omega}{c}x$$

$$\Rightarrow \psi(x,t) = \left[A_1\cos\frac{\omega}{c}x + B_1\sin\frac{\omega}{c}x\right]\cos\omega t + \left[A_3\cos\frac{\omega}{c}x + A_4\sin\frac{\omega}{c}x\right]\sin\omega t$$

FOR FREE END: $\tau = G\frac{\pi a^4}{2}\frac{\delta\psi}{\delta x} = 0$

{FIND CORRESPONDING EIGEN FUNCTION
IN FREE ENDED RODS}

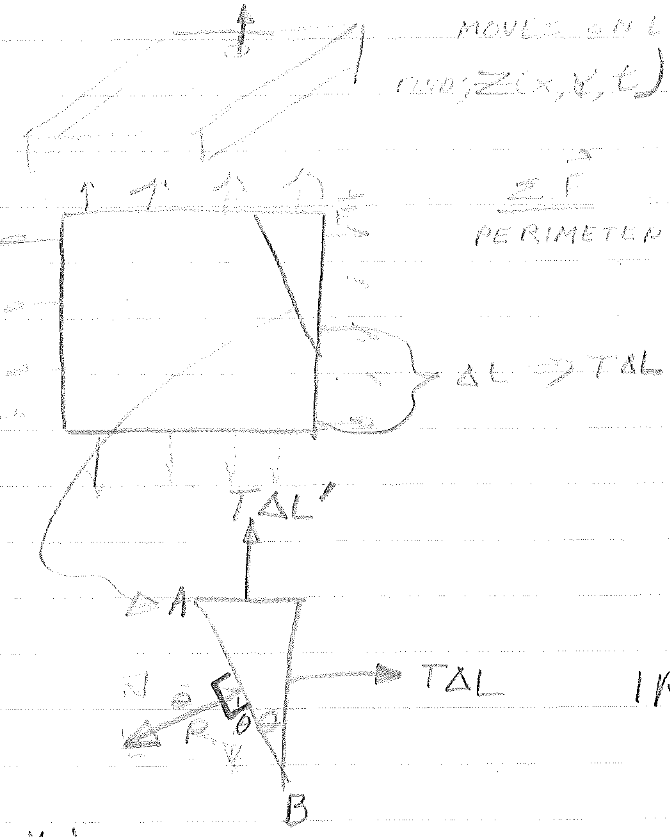
10-11-72

WAVES IN MEMBRANE:

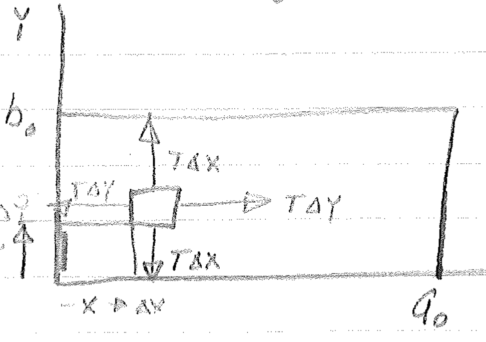
MOVES ONLY IN Z DIRECTION

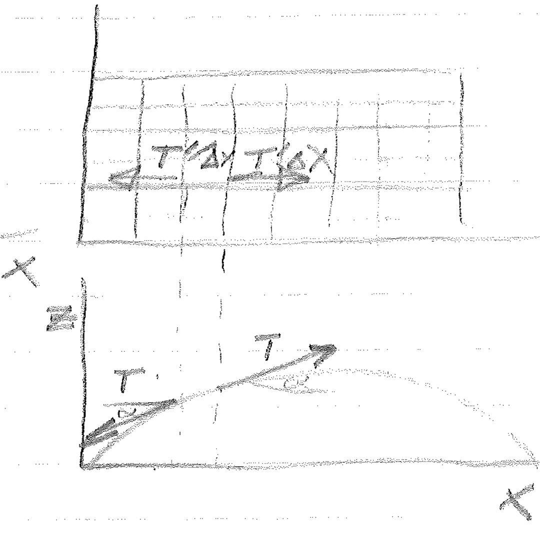
$Z(x,y,t)$

$\frac{\Sigma \vec{F}}{\text{PERIMETER}} = T =$ MEMBRANE LENGTH



$$|R| = \sqrt{(T\Delta L)^2 + T(\Delta L')^2}$$
$$= T\sqrt{\Delta L^2 + \Delta L'^2}$$
$$= T\,L_{AB}$$





$$T' = T'' = T$$

$$T\Delta Y \frac{\delta z}{\delta x}\Big|_{x+\Delta x,\,y,\,t}$$
$$- T\Delta Y \frac{\delta z}{\delta x}\Big|_{x,y,t}$$
$$+ T\Delta X \frac{\delta z}{\delta y}\Big|_{x,y+\Delta y,t}$$
$$- T\Delta X \frac{\delta z}{\delta y}\Big|_{x,y,t}$$
$$= (\sigma\,\Delta X\,\Delta Y)\frac{\delta^2 z}{\delta t^2}\Big|_{X+\frac{\Delta X}{2},\,Y+\frac{\Delta Y}{2},\,t}$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} T\left\{\frac{\frac{\delta z}{\delta x}\Big|_{x+\Delta x,y,t} - \frac{\delta z}{\delta x}\Big|_{x,y,t}}{\Delta x}\right\}$$

$$+\,T\left\{\frac{\frac{\delta z}{\delta y}\Big|_{x,y+\Delta y,t} - \frac{\delta z}{\delta y}\Big|_{x,y,t}}{\Delta t}\right\} = \gamma \frac{\delta^2 z}{\delta t^2}$$

$$\Rightarrow T\left[\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2}\right] = \sigma \frac{\delta^2 z}{\delta t^2}$$

$$\left(\therefore c^2\left[\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2}\right] = \frac{\delta^2 z}{\delta t^2} \Rightarrow c = \sqrt{T/\sigma}\right)$$

$$z(x,y,t) = \mathbb{X}(x)\,\mathbb{Y}(y)\,H(t)$$

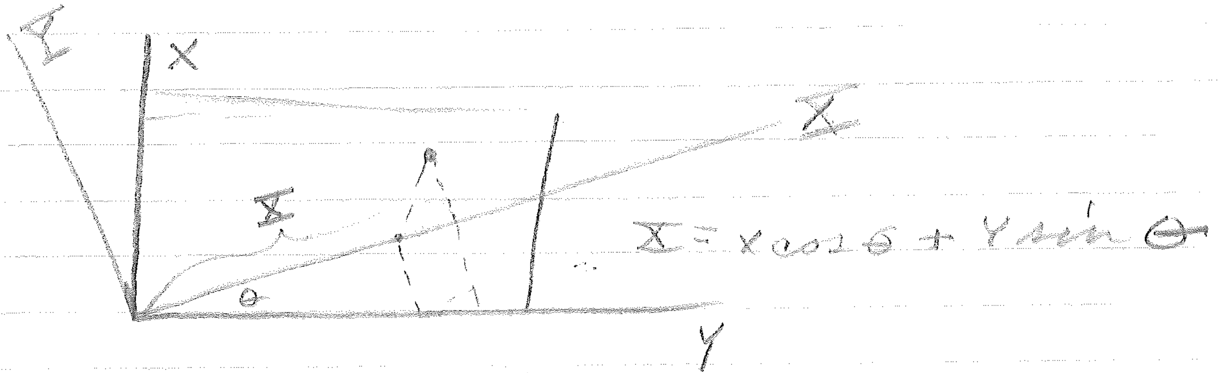
$f(u)$ z $u = ct - (X\cos\theta + Y\sin\theta)$

$\frac{\delta z}{\delta x} = \frac{\delta z}{\delta y}\frac{\delta y}{\delta x} = \frac{\delta z}{\delta u}(-\cos\theta$

$\frac{\delta^2 z}{\delta x^2} = \frac{\delta^2 z}{\delta u^2}(\cos^2\theta)$

$\frac{\delta^2 z}{\delta y^2} = \frac{\delta^2 z}{\delta y^2}\sin^2\theta$

$\frac{\delta^2 z}{\delta t^2} = \frac{\delta^2 z}{\delta y^2}c^2$



$$z = \mathbb{X}(x)\,\mathbb{Y}(y)\,H(t)$$

$$\frac{c^2}{\mathbb{X}}\frac{d^2\mathbb{X}}{dx^2} + \frac{c^2}{\mathbb{Y}}\frac{d^2\mathbb{Y}}{dy^2} = \frac{1}{H}\frac{d^2H}{dt^2} = -\omega^2$$

$$\frac{d^2H}{dt^2} = -\omega^2 H$$

$$\frac{1}{\mathbb{X}}\frac{d^2\mathbb{X}}{dx^2} = -\left(\frac{\omega}{c}\right)^2 - \frac{1}{\mathbb{Y}}\frac{d^2\mathbb{Y}}{dy^2} = -\omega^2$$

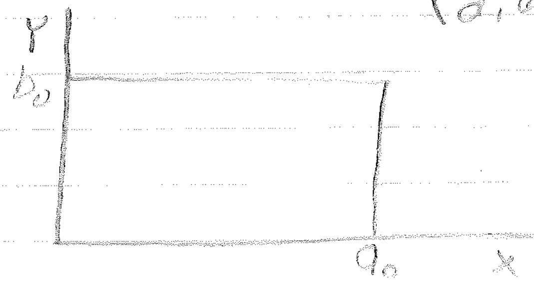
$$\frac{d^2 \overline{X}}{dx^2} = -\alpha^2 H \quad ; \quad \frac{d^2 \overline{Y}}{dY^2} = -\left[\left(\frac{\omega}{c}\right)^2 - \alpha^2\right]\overline{Y} \; ; \; \frac{d^2 H}{dt^2} \quad \omega^2 H$$

$$H = d_1 \cos \omega t + d_2 \sin \omega t$$

$$\overline{X} = d_3 \cos \alpha x + d_4 \sin \alpha X$$

$$\overline{Y} = d_4 \cos \sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\, Y + d_6 \sin \sqrt{\left(\frac{\omega}{c}\right)^2 \alpha^2}\, Y$$

$$\Rightarrow z(x,Y,t) = (d_3 \cos \alpha x + d_4 \sin \alpha x)$$
$$\left(d_5 \cos \sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\, Y + d_6 \sin \sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\, Y\right)$$
$$(d_1 \cos \omega t + d_2 \sin \omega t)$$



BOUNDRY CONDITIONS;

$z(0, Y, t) = 0 \Rightarrow d_3 = 0$

$z(x, 0, t) = 0 \Rightarrow d_5 = 0$

$z(a_0, Y, t) = 0$

$z(X, b_0, t) = 0$

FROM FIRST TWO BOUNDRY CONDITIONS

$$z(x, Y, t) = \left(\sin \alpha x \sin \sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\, Y\right)$$
$$(A \cos \omega t + B \sin \omega t)$$

$0 = z(a_0, Y, t) \Rightarrow \alpha a = m\pi \quad ; m = 1, 2, 3, 4 \ldots$

$0 = z(x, b_0, t) \Rightarrow \sqrt{\left(\frac{\omega}{c}\right) - \left(\frac{m\pi}{a}\right)^2} = n\pi/b; \; n = 1, 2, 3, 4 \ldots$

$$\Downarrow$$

$$\omega_{mn} = c\pi \sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2} \qquad \begin{matrix} n = 1, 2, 3, \\ m = 1, 2, 3, \end{matrix}$$

$$\omega_{11} = c\pi \sqrt{\left(\frac{1}{b}\right)^2 + \left(\frac{1}{a}\right)^2}$$

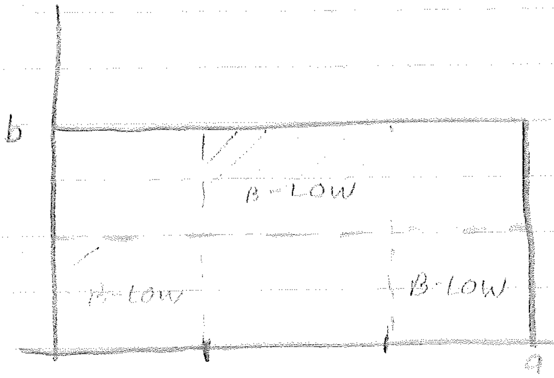
$$\omega_{mn} = c\pi \sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2}$$

$$z_{mn} = \sin \frac{m\pi}{a} X \sin \frac{n\pi}{b} Y \left[A_{mn} \cos c\pi \sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2}\, t\right.$$
$$\left. + B_{mn} \sin c\pi \sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2}\, t\right]$$

$Z_{11} = \sin\frac{\pi}{a}X \sin\frac{\pi}{b}Y \left[C_{11}\cos(\omega_{11}t + \phi_{11})\right]$

$(Z_{11})_{MAX}$ @ $X = \frac{a}{2}$; $Y = \frac{b}{2}$

$Z_{32} = \sin\frac{3\pi}{a}X \sin\frac{2\pi}{b}Y \left[C_{32}\cos\omega_{32}t + \phi_{32}\right]$



$Z(x,y,t) = \sum_{n=1}^{\infty}\sum_{n=2}^{\infty} Z_{mn}(x,y,t)$

GIVEN:

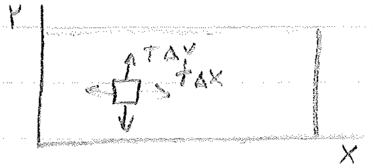
$Z_0(x,Y)$ AND $V_0(x,Y)$

$Z_0(x,Y) = \sum\sum \sin\frac{m\pi}{a}X \sin\frac{n\pi}{b}Y\, A_{mn}$

$V_0(x,Y) = \sum\sum C\pi\sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2}\, \sin\frac{m\pi}{a}X \sin\frac{n\pi}{b}Y B_{mn}$

$\ni \quad A_{mn} = \frac{4}{ab}\int_0^b\int_0^a Z_0(x,Y)\sin\frac{m\pi X}{a}\sin\frac{n\pi Y}{b}\,dx\,dY$

10-12-72



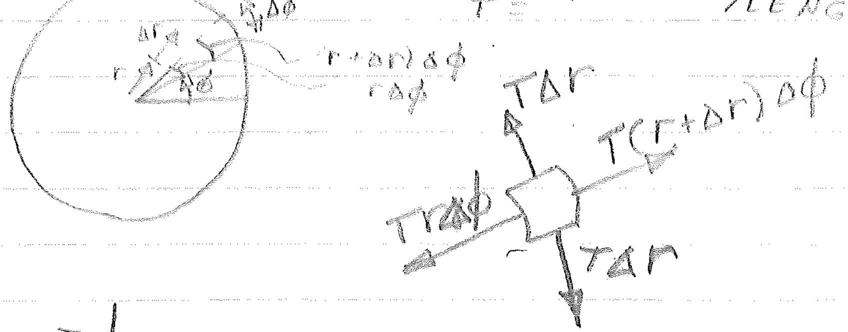
$$c^2\left[\frac{\delta^2 z}{\delta x^2}+\frac{\delta^2 z}{\delta y^2}\right]=\left[\frac{\delta^2 z}{\delta t^2}\right]$$

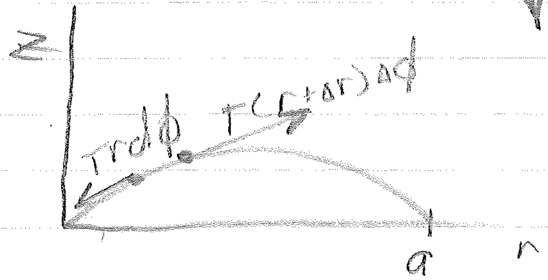
$$c=\sqrt{T/\sigma}$$

$z(x,y,t)=X(x)\,Y(y)\,H(t)$

CIRCULAR MEMBRANE:

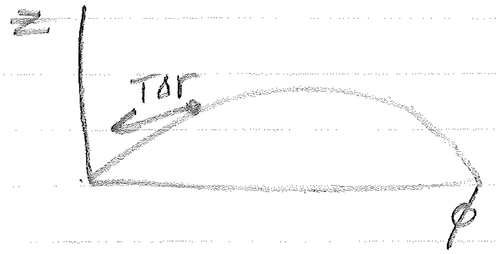
$T =$ FORCE/LENGTH





VERTICAL COMAMONENT SUMMATION IN r

$$T(r+\Delta r)\Delta\phi\,\frac{\delta z}{\delta r}\Big|_{r+\Delta r,\phi,t}-T\,r\,\Delta\phi\,\frac{\delta z}{\delta r}\Big|_{r,\phi,t}$$



DUE TO φ

$$+\,T\Delta r\,\frac{\delta r}{r\delta\phi}\Big|_{r,\phi+\Delta\phi}-T\Delta r\,\frac{\delta r}{\delta\phi}\Big|_{r,\phi,t}$$

$$T(r+\Delta r)\Delta\phi \left.\frac{\delta z}{\delta r}\right|^{\phi, r+\Delta r, t} - T r \Delta\phi \left.\frac{\delta z}{\delta r}\right|^{r,\phi,t} + T\Delta r \left.\frac{\delta z}{r\delta\phi}\right|^{r,\phi+\Delta\phi,t} - T\Delta r \frac{1}{r}\left.\frac{\delta z}{\delta\phi}\right|^{r,t,\phi}$$

$$= \sigma(r\Delta\phi\ \Delta r)\left.\frac{\delta^2 z}{\delta t^2}\right|_{r,\phi,t}$$

TAKING LIMIT AS $\Delta r \to 0$ AND $\Delta\phi \to 0$

$$T\left[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r}\right] + \frac{T}{r^2}\frac{\delta^2 z}{\delta\phi^2} = \sigma\frac{\delta^2 z}{\delta t^2}$$

$$= c^2\left[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r} + \frac{1}{r^2}\frac{\delta^2 z}{\delta\phi^2}\right] = \frac{\delta^2 z}{\delta t^2}$$

LET $z(r,\phi,t) = R(r)\Phi(\phi)H(t)$

$$c^2\left[\Phi H \frac{d^2R}{dr^2} + \frac{1}{r}\phi H\frac{dR}{dr} + \frac{1}{r^2}RH\frac{d^2\Phi}{d\phi^2}\right] = R\Phi\frac{d^2H}{dt^2}$$

$$c^2\left[\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r}\frac{1}{R}\frac{dR}{dt} + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2}\right] = \frac{1}{H}\frac{d^2H}{dt^2} = -\omega^2$$

$$\frac{d^2H}{dt^2} = -\omega^2 H \longleftarrow ①$$

AND:

$$r^2\left[\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{\omega^2}{c^2}\right] = \frac{1}{\phi}\frac{d^2\phi}{d\phi^2} = m^2$$

$$\frac{d^2\phi}{d\phi^2} = -m^2\phi \longleftarrow ② \qquad (k = \omega/c)$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2}\right)R = 0 \longleftarrow ③$$

① $H(t) = d_1 \cos \omega t + d_2 \sin \omega t$

② $\phi(\phi) = d_3 \cos m\phi + d_4 \sin m\phi$

$\Rightarrow z = R(r)\left[d_5 \cos(m\phi + \alpha)\right]\left[d_6 \cos(\omega t + \Omega)\right]$

NOTE $z(r,\phi,t) = z(r,\phi+2n\pi, t)$

$\Rightarrow m = 0, 1, 2, 3, \ldots$

LET $R(r) = \sum_{n=0}^{\infty} a_n r^n$

AGAIN $\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (k^2 - \frac{m^2}{r^2})R = 0$

$-R\frac{m^2}{r^2} = -\frac{m^2}{r^2}a_0 - \frac{m^2}{r}a_1 + m^2 a_2 - m^2 r a_3 - m^2 r^2 a_4 - \dots$

$k^2R = k^2 a_0 + k^2 r a_1 + k^2 r^2 a_2 + k^2 r^3 a_3$

$\frac{1}{r}\frac{dR}{dr} = \frac{a_1}{r} + 2a_2 + 3ra_3 + 4r^3a_4 + 5r^3a_5 + \dots$

$\frac{d^2R}{dr^2} = 2a^2 + 6ra_3 + 12r^2a_4 + 20r^2a_5 \dots$

$$\Rightarrow \frac{d^2R}{dt} = \frac{-m^2}{r^2}a_0 + (1-m^2)\frac{a_1}{r} + (a-m^2)a_2$$
$$+[((4-m^2)a_2) + k^2a_0] + [(9-m^2)a_3 + k^2a_1]r$$
$$+ [(16-m^2)a^4 + k^2a_2]r^2 = 0$$

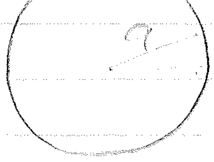
$a_1 = 0$

$a_2 = -\frac{k^2}{4}a_0$

$a_4 = -\frac{k^2}{16}a_2 = \frac{k^4}{(4)(16)}a_0$

$$\Rightarrow R(r) = a_0 - \frac{k^2}{4}a_2 r^2 + \frac{k^2}{(4)(16)}a_0 r^4 + \dots$$

$$= a_0\left[1 - \frac{kr}{4}^2 + \frac{(kr)^4}{(4)(16)} + \dots\right]$$



$$c^2\left[\frac{d^2z}{dr^2}+\frac{1}{r}\frac{\delta z}{\delta r}+\frac{1}{r^2}\frac{\delta^2 z}{\delta\phi^2}\right]=c^2\frac{\delta^2 z}{\delta t^2}$$

$$z(r,\phi,t)=J_m\left(\frac{\omega}{c}r\right)\left[d_3\cos m\phi+d_4\sin m\phi\right]$$
$$\left[d_1\cos\omega t+d_2\sin\omega t\right]$$

$z(a,\phi,t)=0$; $J_m\left(\frac{\omega}{c}a\right)=0$

$\frac{\omega a}{c}=2.405, 5.52, 8.65$

$\omega=\frac{2.405c}{a}, \frac{5.52c}{a}, \frac{8.65c}{a},$

$m=1$ $J_1\left(\frac{\omega}{c}a\right)=0$

$\frac{\omega}{c}a=3.83, 7.01$

$m=2$ $J_2\left(\frac{\omega}{c}a\right)=0$

$\frac{\omega}{c}a=5.13, 8.41$

$$z_{01}=C_{01}J_0\left(\frac{2.405}{a}r\right)\cos\left(\frac{2.405c}{a}t+\Omega_{01}\right)$$

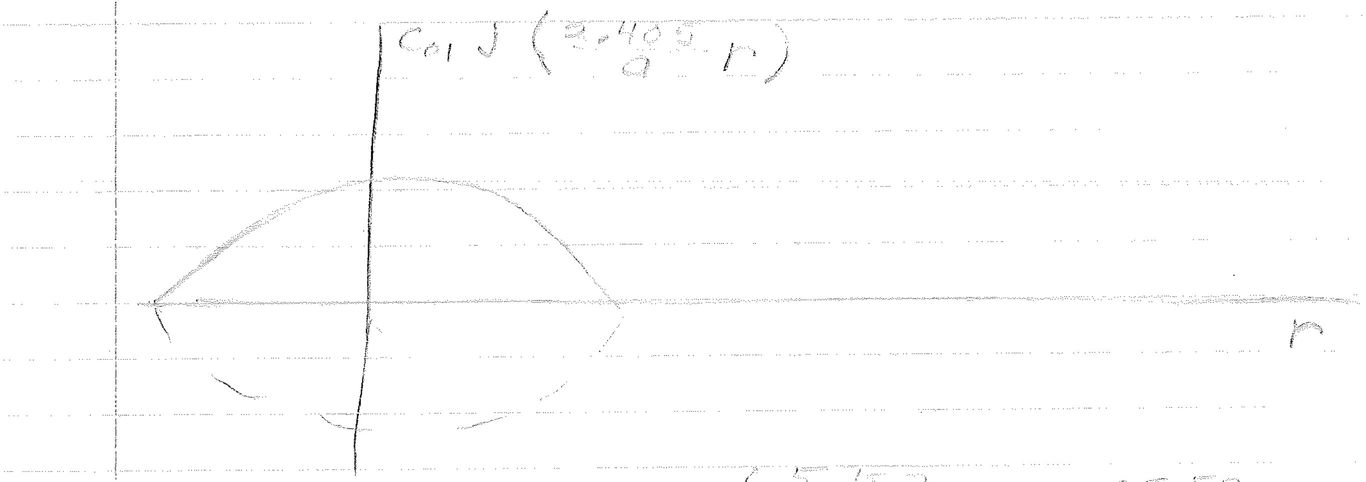
$$z_{02}=C_{02}J_0\left(\frac{5.52}{a}r\right)\cos\left(\frac{5.52}{a}ct+\Omega_{02}\right)$$

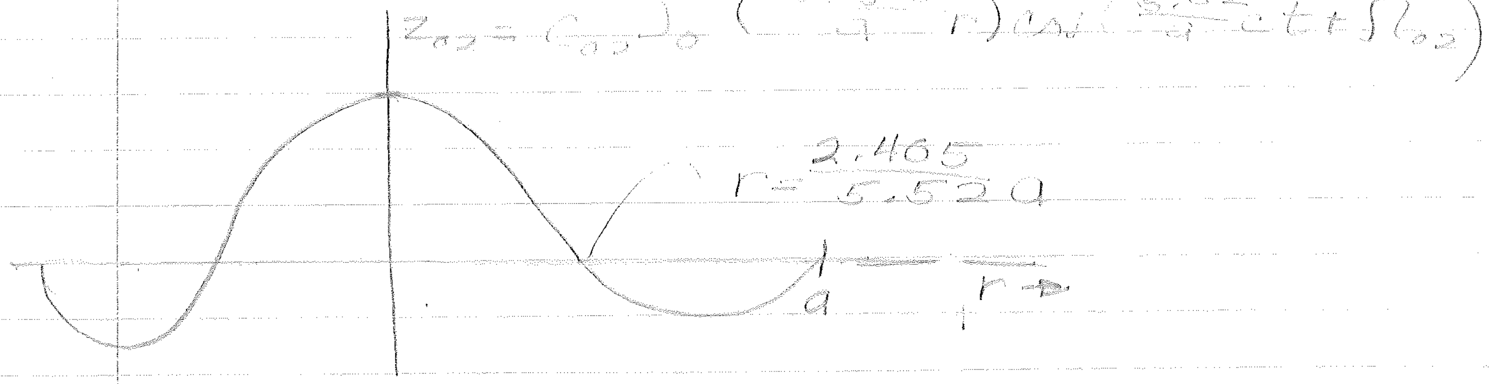
⋮

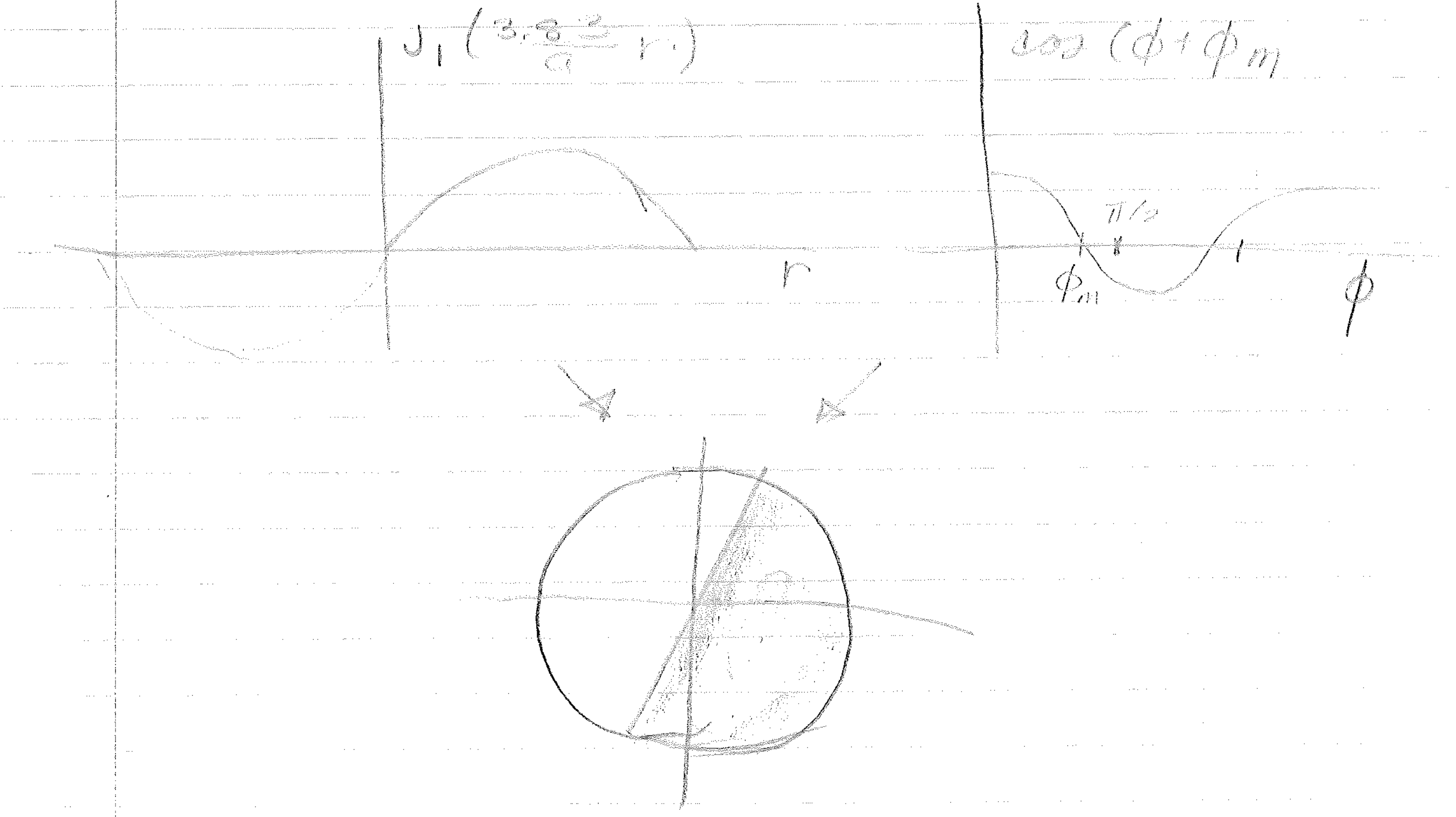
$$z_{11}=C_{11}J_1\left(\frac{3.81}{a}r\right)\cos\left[\phi+\phi_{11}\right]\left[\cos\frac{3.83c}{a}t+\Omega_{11}\right]$$
$$z_{12}=C_{12}J_2\left(\frac{7.01}{a}r\right]\cos(\phi+\phi_{12})\left[\cos\omega_{12}t+\Omega_{11}\right)$$

$$z_{21}=C_{21}J_2\left(\frac{5.13}{a}r\right)\cos(2\phi+\phi_{21})(\cos\omega_{21}t+\Omega_{21})$$

FUNDEMENTAL MODE







4-5 4-9 4-10 SUP. PROB. 2,3,4,5 (TEST NEXT THURS)

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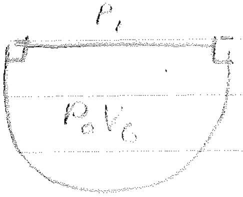
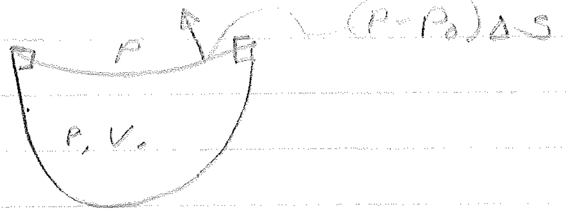
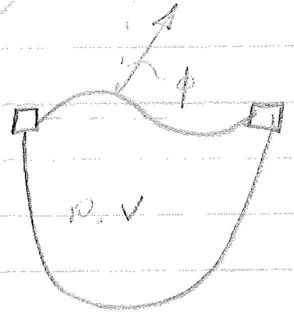
$$z(r,\phi,t) = J_m\left(\tfrac{\omega}{c}r\right)\left[d_3 \cos m\phi + d_4 \sin m\phi\right]$$
$$\left[d_1 \cos \omega t + d_2 \sin \omega t\right.$$

EIGEN FREQ. FROM $J_m\left(\frac{\omega}{c}a\right)=0$







ASSUME $PV^{\gamma} =$ CONSTANT

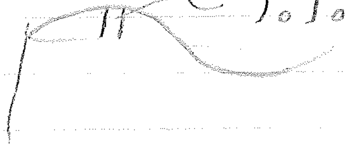
$$[\text{TENSILE FORCES}] + [P-P_0]\Delta S = \sigma \Delta S \frac{\delta^2 z}{\delta t^2}$$

$$c^2\left[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r} + \frac{1}{r^2}\frac{\delta^2 z}{\delta\phi^2}\right] + \frac{P-P_0}{\sigma} = \delta^2 z/\delta t^2$$

$$\Delta P_0 = dP = \frac{-\gamma P_0}{V_0} dV$$

$$\Rightarrow c^2\left[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r} + \frac{1}{r^2}\frac{\delta^2 z}{\delta\phi^2}\right] - \frac{\gamma P_0}{V_0} dV = \sigma \Delta S \frac{\delta^2 z}{\delta t^2}$$

$$\int_0^{2\pi}\int_0^{a} z(r,\phi,t)\, r\,dr\,d\phi = dV$$



SOLUTION:

$$z(r,\phi,t) = \psi(r,\phi)\,H(t)$$

$$dV = \int_0^{2\pi}\int_0^a \psi(r,\phi)\,H(t)\,r\,dr\,d\phi$$

$$H I_0 \ni \quad I_0 = \int_0^{2\pi}\int_0^a r\,\psi(r,\phi)\,dr\,d\phi$$

$$c^2\left[H\frac{\delta^2\psi}{\delta r^2} + \frac{H}{r}\frac{\delta\psi}{\delta r} + \frac{H}{r^2}\frac{\delta^2\psi}{\delta\phi^2}\right] - \frac{\delta P_0 I_0 H}{V_0\sigma} = \psi\,\delta^2 H/\delta t^2$$

$$\frac{c^2}{\psi}\left[\frac{\delta^2\psi}{\delta r^2} + \frac{1}{r}\frac{\delta\psi}{\delta r} + \frac{1}{r^2}\frac{\delta^2\psi}{\delta\phi^2}\right] - \frac{\delta P_0 I_0}{V_0\sigma\psi} = \frac{1}{H}\frac{\delta^2 H}{\delta t^2} = -\omega^2$$

FOR $\frac{\delta^2 H}{\delta t^2} = -\omega^2 H \quad H = d_1\cos\omega t + d_2\sin\omega t$

FOR $\frac{c^2}{\psi}\left[\frac{\delta^2\psi}{\delta r^2} + \frac{1}{r}\frac{\delta\psi}{\delta r} + \frac{1}{r^2}\frac{\delta^2\psi}{\delta\phi^2}\right] + \omega^2 = \delta P_0 I_0 / V_0\sigma\psi$

$$\left[\frac{\delta^2\psi}{\delta r^2} + \frac{1}{r}\frac{\delta\psi}{\delta r} + \frac{1}{r^2}\frac{\delta^2\psi}{\delta\phi^2}\right] + k^2\psi = \delta P_0 I_0 / V_0\sigma c^2$$

LET $\psi(r,\phi) = R(r) + \Phi(\phi)$

$$\Rightarrow \frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi} + k^2 = 0$$

$$r^2\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + k^2R\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2$$

$$\psi(r,\phi) = J_m(kr)\left[d_3\cos m\phi + d_4\sin m\phi\right] + \frac{\delta P_0 I_0}{V_0\sigma c^2 k}$$

$$\Rightarrow z(r,\phi,t) = \left[J_m(kr)\{d_3\cos m\phi + d_4\cos m\phi\} + \delta P_0 I_0 / V_0\sigma c^2 k\}\right]\left[d_1\cos\omega t + d_2\sin\omega t\right]$$

$$I_0 = \int_0^a\int_0^{2\pi}\left[J_m(kr)\left[d_3\cos m\phi + d_4\sin\phi\right] + \delta P_0 I_0 / V_0\sigma c^2 k^2\right] r\,dr\,d\phi$$

$$I_0\left[1 - \frac{\sigma P_0\pi a^2}{V_0\sigma c^2 k^2}\right] = \int_0^a\int_0^{2\pi} J_m(kr)(d_3\cos m\phi + d_4\sin m\phi)\,r\,dr\,d\phi$$

IF $m \neq 0$, $I_0 = 0$

IF $m = 0$,

$$I_0 = \frac{d_3}{\left[1 - \frac{\delta P_0\pi a^2}{V_0\sigma c^2 k^2}\right]} = \int_0^a\int_0^{2\pi} J_m(kr)\,a\,r\,d\phi$$

$$= \left[\frac{2\pi}{1 - \frac{\delta P_0\pi a^2}{V_0\sigma c^2 k^2}}\right]\frac{(kr)J_1(kr)}{k^2}\Bigg|_0^a = \left[\frac{2\pi/a}{1 - \frac{\sigma P_0\pi a^2}{V_0\sigma c^2 k^2}}\right]\sigma_1(k_a)$$

$$z(r,\phi,t) = \left[J_0(kr) + \frac{\gamma P}{V_0 \sigma \omega^2}\left[\frac{\pi 2 a^2}{1 - \frac{\gamma P_0 \pi a^2}{V_0 \sigma c^2 k}}\right]\frac{J_1(k_a)}{k_a}\right] \cdot (A \cos \omega t + B \sin \omega T)$$

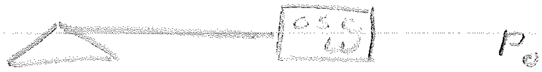
$z(a,\phi,t) = 0$

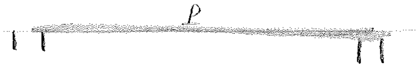
$\Rightarrow J_0(ka) = -\alpha\, J_1(ka)/(ka)^2$

$\Rightarrow \alpha = \gamma P_0 \pi a^4 / T V_0 \Rightarrow T = \text{TENSION}$

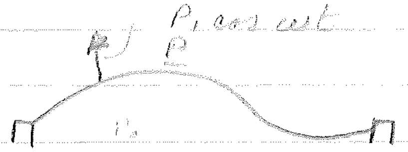
TEST ON MONDAY

DRIVEN MEMBRANE





$\underline{p} = P_0 + P_1 \cos \omega t$



$[\text{TENSILE FORCES}] + P_1 \cos \omega t \, \Delta S = \sigma\, \delta^2 z / \delta t^2$

$$T\left[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r} + \frac{1}{r^2}\frac{\delta^2 z}{\delta \phi^2}\right] + \frac{P_1}{\sigma}\cos \omega t = \delta^2 z/\delta t^2$$

ASSUME $z(r,\phi,t) = \psi(\phi,r)\, H(t)$

$$c^2\left[\frac{\delta^2 \psi}{\delta r^2} + \frac{1}{r}\frac{\delta \psi}{\delta r} + \frac{1}{r^2}\frac{\delta^2 \psi}{\delta \phi^2}\right]\cos \omega t + \frac{P_1}{\sigma}\cos \omega t = -\omega^2 \psi \cos \omega t$$

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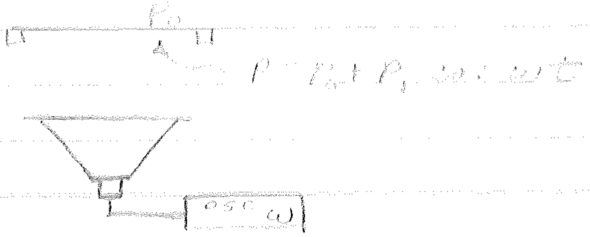


$$[\text{TENSILE FORCES}] + P_1 \cos \omega t \, \Delta S = \sigma \Delta S \frac{\partial^2 z}{\partial t^2}$$

$$c^2\left[\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2}\right] + \frac{P_1}{\sigma} \cos \omega t = \frac{\partial^2 z}{\partial t^2}$$

$$z(r,\phi,t) = \psi(r,\phi) \cos \omega t$$

$$\Rightarrow c^2\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2}\right] \cos \omega t + \frac{P_1}{\sigma} \cos \omega t = \psi(-\omega^2) \cos \omega t$$

$$\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2}\right] + k^2 \psi = \frac{-P_1}{\sigma c^2}$$

IF $\frac{P_1}{\sigma c^2} = 0$; $\psi(r,\phi) = J_m(kr)[d_3 \cos m\phi + d_4 \sin m\phi]$

IF $\frac{P_1}{\sigma c^2} \neq 0$; $\psi(r,\phi) = J_m(kr)[d_3 \cos m\phi + d_4 \sin m\phi) - \frac{P_1}{\sigma c^2 k^2}$

$$\Rightarrow z(r,\phi,t) = \left([J_m(kr)][d_3 \cos m\phi + d_4 \sin m\phi] - \frac{P_1}{\sigma k^2 c^2}\right) \cos \omega t$$

BOUNDARY CONDITIONS

$$J_m(ka)[d_3 \cos m\phi + d_4 \sin m\phi] - \frac{P_1}{\sigma c^2 k^2} = 0$$

$$\Rightarrow d_3 \cos m\phi + d_4 \sin m\phi = \frac{P_1}{\sigma c^2 k^2 J_m(ka)}$$

$$\Rightarrow m = 0$$

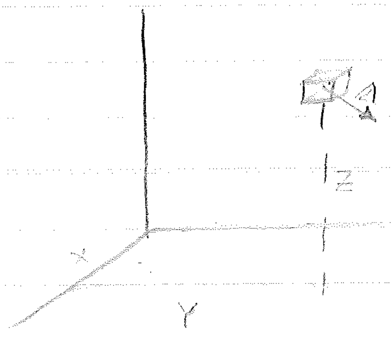
$$\therefore d_3 = \frac{P_1}{\sigma c^2 k^2 J_0(ka)}$$

$$\Rightarrow z(r,t) = \left\{J_0(kr)\left[\frac{P_1}{\sigma c^2 k^2 J_0(ka)}\right] - \frac{P_1}{\sigma c^2 k^2}\right\} \cos \omega t$$

$$= \frac{P_0}{\sigma \omega^2}\left[\frac{J_0(kr)}{J_0(ka)} - 1\right] \cos \omega t \;;$$

FOR $J_0(ka) = J_0(\frac{\omega}{c} a) = 0$, $z(r,t) = \infty$

WAVES IN FLUIDS

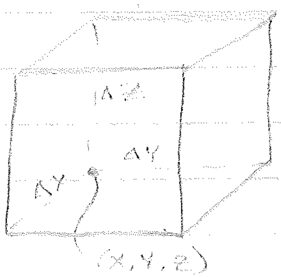


DISPLACEMENTS

ξ, η, ζ

STRAIN

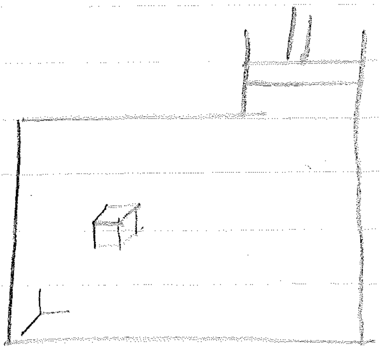
$\frac{\delta \xi}{\delta x}, \frac{\delta \eta}{\delta y}, \frac{\delta \zeta}{\delta z}$



IN AN IDEAL FLUID, THERE ARE NO SHEARING STRAIN:

$$\Rightarrow \Delta P = -B\left[\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta Y} + \frac{\delta \zeta}{\delta z}\right]$$ ← STRESS STRAIN RELATIONSHIP

$$= -B(\epsilon_{xx} + \epsilon_{YY} + \epsilon_{zz})$$

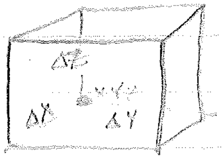


$$P' - P_0 = -B\left(\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta Y} + \frac{\delta \zeta}{\delta z}\right)$$

$$p \triangleq P' - P_0$$ = ACOUSTIC PRESSURE

$$p = -B\left(\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta Y} + \frac{\delta \zeta}{\delta z}\right)$$

WAVES



ON FRONT AND BACK FACES:

$$P'(x,y,z)\Delta Y \Delta Z - P'(x+\Delta x, Y, z)\Delta Y \Delta Z = \rho \Delta x \Delta Y \Delta Z \frac{\delta^2 \xi}{\delta t^2}$$

$$\Rightarrow -\frac{\delta P'}{\delta Z} = \rho \frac{\delta^2 \xi}{\delta t^2}$$

SIMILARLY : $-\frac{\delta P'}{\delta Y} = \rho \frac{\delta^2 \eta}{\delta t^2}$

: $-\frac{\delta P'}{\delta Y} = \rho \frac{\delta^2 \zeta}{\delta t^2}$

OR $-\overrightarrow{\text{grad}}\, p' = \rho \frac{\delta^2 \vec{d}}{\delta t^2} = \rho \frac{\delta \vec{v}}{\delta t} \ni v = \frac{\delta \vec{d}}{\delta t}$

$\text{grad}\, \wp = \rho$ [illegible]

RECALL:

$\wp' = -B\left[\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta y} + \frac{\delta \zeta}{\delta z}\right]$

$\frac{\delta^2 p}{\delta t^2} = -B\left[\frac{\delta}{\delta x}\frac{\delta^2 \xi}{\delta t^2} + \frac{\delta}{\delta y}\frac{\delta^2 \eta}{\delta t^2} + \frac{\delta}{\delta z}\frac{\delta^2 \zeta}{\delta t^2}\right]$

$\Rightarrow \frac{\delta^2 \wp}{\delta t^2} = \frac{B}{\rho}\left[\frac{\delta^2 \wp}{\delta x^2} + \frac{\delta^2 \wp}{\delta y^2} + \frac{\delta^2 \wp}{\delta z^2}\right]$

$= c^2\left[\frac{\delta^2 \wp}{\delta x^2} + \frac{\delta^2 \wp}{\delta y^2} + \frac{\delta^2 \wp}{\delta z^2}\right]$

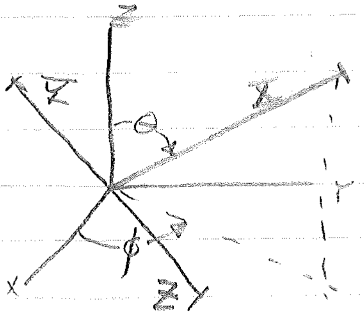
$c = \sqrt{B/\rho} \ni B$ = ADIABATIC BULK MODULUS

"VELOCITY OF PLANE WAVES

SOLUTIONS:

$\rho = Ae^{i(\omega t - kr)}$ (PLANE WAVE)

$f(v) \ni v = ct - \underbrace{(x \sin\theta\cos\phi + Y\cos\theta\sin\phi + z\cos\theta)}_{X}$



FOR IDEAL GAS: $B_A = \gamma P \Rightarrow c = \sqrt{\gamma P/\rho}$

$PV = nRT$ $V = \frac{m}{\rho} = nM/\rho$

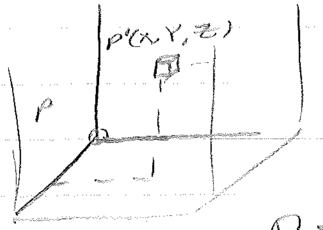
$V = nRT/V = \frac{nm}{\rho}$

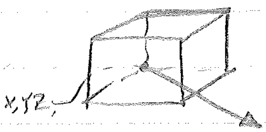
$\frac{P}{\rho} = RT/m$

$\Rightarrow c = \sqrt{\gamma RT/m} = \text{const}\sqrt{T}$

10-19-72

WAVES IN FLUIDS





$\vec{s} = (\xi, \eta, \zeta)$

DISTORTIONS MEASURED: $\left(\frac{\delta \xi}{\delta x}, \frac{\delta \eta}{\delta y}, \frac{\delta \zeta}{\delta z}\right)$

$$P = p' - p$$

$$P = -B\left[\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta y} + \frac{\delta \zeta}{\delta z}\right]$$

$$\Rightarrow -\overrightarrow{grad}\, P = \rho \frac{\delta^2 s}{\delta t^2} \qquad \left\{ -\frac{\delta P}{\delta x} = \rho \frac{\delta^2 \xi}{\delta x^2}\right.$$

$$\therefore c^2\left[\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta y^2} + \frac{\delta^2 P}{\delta z^2}\right] = \frac{\delta^2 P}{\delta t^2} \qquad \ni c = \sqrt{B/\rho}$$

$$P(x,y,z,t) = X(x)\,Y(y)\,Z(z)\,H(t)$$

YIELDS:

$$c^2\left[\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2}\right] = \frac{1}{H}\frac{d^2 H}{dt^2} = -\omega^2$$

$$\frac{d^2 H}{dt^2} = -\omega^2 H \Rightarrow$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2} - \frac{1}{Z}\frac{d^2 Z}{dz^2} - k^2 = -\alpha^2$$

$$\frac{d^2 X}{dx^2} = -\alpha^2 X \Rightarrow X = a_1 \cos \alpha x + b_1 \sin \alpha x$$

$$\frac{1}{Y}\frac{d^2 Y}{dy^2} = -\alpha^2 + k^2 + \frac{1}{Z}\frac{d^2 Z}{dz^2} = -\beta^2$$

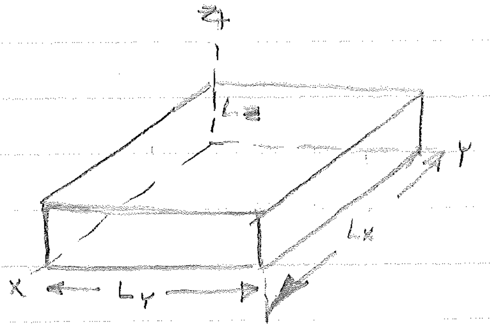
$\frac{d^2 Y}{dy^2} = -\beta^2 Y$ ~~Y~~ $a_2 \cos \beta Y + b_2 \sin \beta Y$

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} = -\sqrt{k^2 - \alpha^2 - \beta^2} = \gamma^2 \Rightarrow Z = a_3 \cos \gamma + b_3 \sin \gamma$$

$P(x,y,z,t) = [a_1 \cos \alpha x + b_1 \sin \alpha x]$
$[a_2 \cos \beta Y + b_2 \sin \beta Y]$
$[a_3 \cos \gamma$ ~~Z~~ $+ b_3 \sin \gamma$ ~~Z~~$]$
$[a_4 \cos \omega t + b_4 \sin \omega t]$

$$= A \cos(\alpha x + \Omega_1)\cos(\beta Y + \Omega_2)\cos(\gamma X + \Omega_3)\cos(\omega t + \Omega_4)$$

(DETERMINE AS FUNCTION OF 8 PLANE WAVES)



$$\xi(0,Y,Z,t)=0$$
$$\Rightarrow \frac{\delta\xi}{\delta t}\Big|_{0YZt}=0$$
$$\frac{\delta^2\xi}{\delta t}\Big|_{0YZt}=0$$

$$\Rightarrow \frac{\delta P}{\delta X}\Big|_{0,Y,Z,t}=0 \Rightarrow b_1=0$$

$$\eta(x,0,z,t)=0$$
$$\Rightarrow \frac{\delta\eta}{\delta t}\Big|_{0YZt}=0$$
$$\Rightarrow \frac{\delta P}{\delta Y}\Big|_{X,0,Z,t}=0 \Rightarrow b_2=0$$

SIMILARLY

$$\frac{\delta P}{\delta Z}\Big|_{X,Y,0,t}=0 \Rightarrow b_3=0$$

REDUCING THE EQUATION:

$$P(x,Y,Z,t)=(a_1 a_2 a_3)\cos\alpha X\cos\beta Y\cos\gamma Z$$
$$[a_4\cos\omega t+b_4\sin\omega t]$$

$$0=\xi(L_x,Y,Z,t)\Rightarrow\frac{\delta P}{\delta X}\Big|_{L_X,Y,Z,t}=0\Rightarrow\alpha={}^{n_X\pi}/_{L_X}$$

$$0=\eta(x,L_Y,Z,t)\Rightarrow\frac{\delta P}{\delta Y}\Big|_{X,L_Y,Z,t}=0\Rightarrow\beta={}^{n_Y\pi}/_{L_Y}$$

$$0=\zeta(X,Y,L_Z,t)\Rightarrow\frac{\delta P}{\delta Z}\Big|_{X,Y,L_Z,t}=0\Rightarrow\gamma={}^{n_Z\pi}/_{L_Z}$$

$$\gamma^2=k^2-\alpha^2-\beta^2$$
$$=\left(\frac{\omega}{c}\right)^2-\alpha^2-\beta^2\Rightarrow\left(\frac{\omega}{c}\right)^2=\gamma^2+\alpha^2+\beta^2$$
$$=\left(\frac{n_X\pi}{L_X}\right)^2+\left(\frac{n_Y\pi}{L_Y}\right)^2+\left(\frac{n_Z\pi}{L_Z}\right)^2$$
$$\therefore\ \omega_{n_X n_Y n_Z}=\pi c\sqrt{\left(\frac{n_X}{L_X}\right)^2+\left(\frac{n_Y}{L_Y}\right)^2+\left(\frac{n_Z}{L_Z}\right)^2}$$

$$P_{n_X,n_Y,n_Z}=\cos\frac{n_X\pi}{L_X}X\cos\frac{n_Y\pi}{L_Y}Y\cos\frac{n_Z\pi}{L_Z}$$
$$[A_{n_X n_Y n_Z}\cos\omega_{n_X n_Y n_Z}t+B_{n_X n_Y n_Z}\sin\omega_{n_X n_Y n_Z}t]$$

IF $L_X>L_Y>L_Z$; FUND FREQ. WOULD BE $\frac{\pi c}{L_X}$

$$P_{100}=\cos\frac{\pi}{L_X}X\left[A_{100}\cos\frac{\pi}{L_X}ct+B_{100}\cos\frac{\pi}{L_X}ct\right]$$
$$=C_{100}\cos\frac{\pi}{L}X\left[\cos\left(\frac{\pi}{L_X}ct+\Omega_{100}\right)\right]$$

$$-\text{grad}\, P = \rho \frac{\delta^2 \bar{A}}{\delta t^2}$$

$$-\frac{\delta P}{\delta x} = \rho \frac{\delta^2 \xi}{\delta t^2}$$

$$\left.\begin{aligned} -\frac{\delta P}{\delta y} &= \rho \frac{\delta^2 \eta}{\delta t^2} \\ -\frac{\delta P}{\delta z} &= \rho \frac{\delta^2 \zeta}{\delta t^2} \end{aligned}\right\} = 0$$

$$-\frac{\delta P_{100}}{\delta x} = C_{100} \frac{\pi}{L_x} \sin \frac{\pi}{L} x \left[\cos(\omega_{100} t + \Omega_{100})\right] = \rho \frac{\delta^2 \xi}{\delta t^2}$$

$$\Rightarrow \frac{\delta^2 \xi}{\delta t^2} = C_{100} \rho \frac{\pi}{L_x} \sin \frac{\pi}{L} x \left[(\cos \omega_{100} t + \Omega_{100})\right]$$

$$\xi = C_{100} \rho \frac{\pi}{L_x \omega_{100}^2} \cos \frac{\pi}{L} x \left[(\sin(\omega_{100} t + \Omega_{100})\right]$$

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2, 3, 5, 6, 8, 9, 10, 11

$$P = -B_0 \left[\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta y} + \frac{\delta \zeta}{\delta z}\right] = -B_0 \,\text{div}\, \vec{S}$$

~~$\Rightarrow -\text{grad}\, P = \rho\, \delta^2\xi/\delta t^2$~~

$$c^2\left(\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta y^2} + \frac{\delta^2 P}{\delta z^2}\right) = \frac{\delta^2 P}{\delta t^2}$$

$$P(x,t) = f_1(x - ct)$$

$$P(y,t) = f_2(y - ct)$$

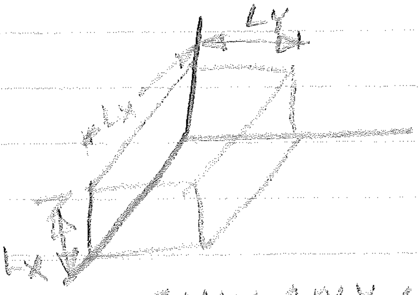
$$P(z,t) = f_3(z - ct$$

on $P(v) \Rightarrow v = ct - \left[x \cos\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta\right]$

$$P(x,y,z,t) = X(x)\, Y(y)\, Z(z)\, H(t)$$

$$\begin{aligned} \Rightarrow P(x,y,z,t) = \; & (a_1 \cos \alpha x + b_1 \sin \alpha x) \\ & (a_2 \cos \beta y + b_2 \sin \beta y) \\ & (a_3 \cos \gamma z + b_3 \cos \gamma z) \\ & (a_4 \cos \omega t + b_4 \sin \omega t) \end{aligned}$$

$$\alpha^2 + \beta^2 + \gamma^2 = \left(\frac{\omega}{c}\right)^2$$



$$\xi(0,Y,Z,t)=0 \Rightarrow \frac{\delta\rho}{\delta x}\Big|_{0,Y,Z,t}=0$$

$$\xi(L_x,Y,Z,t)=0 \Rightarrow \frac{\delta\rho}{\delta x}\Big|_{L_x,Y,Z,t}=0$$

$$\Rightarrow b_1=0 ;$$

$$\alpha = n_x\pi/L_x$$

SIMILARLY;

$$b_2=0 \quad ; \quad \beta = \frac{n_Y\pi}{L_Y}$$

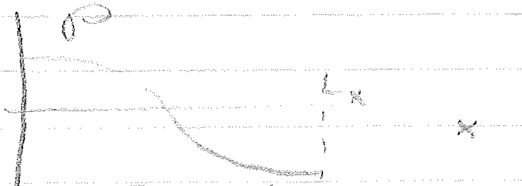
$$b_3=0 \quad ; \quad \gamma = n_z\pi/L_z$$

$$\Rightarrow \left(\frac{\omega}{c}\right) = \pi\sqrt{\left(\frac{n_x}{L_x}\right)^2+\left(\frac{n_Y}{L_Y}\right)^2+\left(\frac{n_z}{L_z}\right)^2}$$

$$\Rightarrow \rho_{n_x n_Y n_z}(x,Y,z,t) = \cos\left(\frac{n_x\pi}{L_x}\right)x \cos\left(\frac{n_Y\pi}{L_Y}\right)Y \cos\frac{n_z\pi}{L_z}z\,\Big[A_{n_x n_Y n_z}\cos\omega_{n_x n_Y n_z}t + B_{n_x n_Y n_z}\sin\omega_{n_x n_Y n_z}t\Big]$$

$$\omega_{100}=\frac{\pi c}{L_x}$$

$$\rho(x,t) = C_{100}\cos\frac{\pi}{L_x}X\cos(\omega_{100}t+\Omega_{100})$$

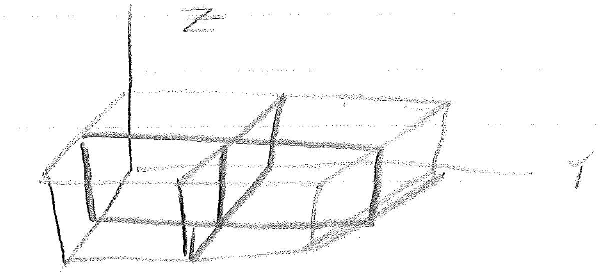


$$\rho(x,t) = C_{100}\,\tfrac{1}{2}\Big[\cos\left(\tfrac{\pi}{L_x}X+\omega_{100}t+\Omega\right) + \cos\left(\tfrac{\pi}{L_x}X-\omega_{100}t-\Omega\right)\Big]$$

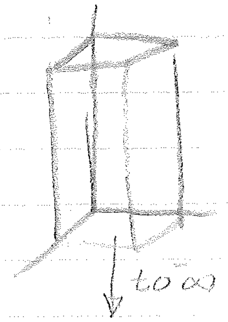
SIMILARLY

$$\rho(Y,t) = C_{010}\cos\frac{\pi}{L_Y}Y\cos(\omega_{010}t+\Omega_{010})$$

$$\rho_{110}(x,Y,t) = C_{110}\cos\frac{\pi}{L_x}X\cos\frac{\pi}{L_Y}Y\cos(\omega_{110}t+\Omega_{110})$$



WAVEGUIDE



$$\rho(x,y,z,t) = (a_1 \cos \alpha x + b_1 \sin \alpha x)$$
$$(a_2 \cos \beta x + b_2 \sin \beta x)$$
$$(a_3 \cos \gamma x + b_3 \sin \gamma x)$$
$$(a_4 \cos \omega t + b_4 \cos \omega t)$$

$$\rho_{0,y,z,t} = \rho_{L_x,y,z,t} = 0$$
$$\rho_{x,0,z,t} = \rho_{x,L_y,z,t} = 0$$

Ⓐ

$$\Rightarrow \rho(x,y,z,t) = C \cos \frac{n_x \pi}{L_x} X \cos \frac{n_y \pi}{L_y} Y$$
$$\cos(\gamma z + \delta) \cos(\omega t + \Omega)$$

$$\rho(x_0,y_0,z,t) = A' \cos(\gamma z + \delta) \cos(\omega t + \Omega)$$
$$= \frac{A}{2}\left[\cos(\gamma z + \omega t + \Omega) + \cos(\gamma z + \delta - \omega t - \Omega\right]$$

$$= \frac{A'}{2}\left[\cos \gamma\left[z + \frac{\omega}{\gamma} t + \frac{\Omega + \delta}{\gamma}\right) + \cos \gamma\left(z - \frac{\omega}{\gamma} t + \frac{\delta - \Omega}{\gamma}\right)\right]$$
$$\hookrightarrow c' = \omega / \gamma$$

$$c' = \frac{\omega}{\gamma} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n_x \pi}{L_x}\right)^2 - \left(\frac{n_y \pi}{L_y}\right)^2}} = \frac{c}{\sqrt{\left(1 - \left(\frac{n_x \pi}{L_x} + \frac{n_y \pi}{L_y}\right)^2\right)\frac{c^2}{\omega^2}}}$$

FOR FIXED ω, THERE IS A SOLUTION OF Ⓐ OF EVERY VALUE OF n_x & n_y FOR WHICH:

$$\text{Ⓑ} \quad \left(\frac{\omega}{c}\right)^2 > \left(\frac{n_x \pi}{L_x}\right)^2 + \left(\frac{n_y \pi}{L_y}\right)^2$$

$n_x = n_y = 0 \Rightarrow$ 0,0 MODE

$$\rho_{00}(\cancel{X},\cancel{Y},z,t) = C_{00} \cos(\gamma z + \delta) \cos(\omega t + \Omega)$$

TWO PLANE WAVES

$$\rho_{10}(y,z,t) = C_{01} \cos \frac{\pi}{L_x} X \cos(\gamma z + \delta) \cos(\omega t + \Omega)$$

$$c' = \frac{c}{\sqrt{1 - \left(\frac{\pi}{L_x}\right)^2 \left(\frac{c^2}{\omega^2}\right)}}$$

MAY INSURE (0,0) MODE BY INSURING Ⓑ

PLANE WAVES:

$$P = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)} \ni k = \omega/c$$

$$-\frac{\delta P}{\delta x} = \rho\, \delta^2\xi/\delta t^2$$

$$\frac{\delta P}{\delta x} = -[A(-ik)e^{i(\omega t - kx)} + B(ik)e^{i(\omega t + kx)}]$$

$$\Rightarrow \xi = \frac{A(-ik)}{\rho(\omega^2)} e^{i(\omega t - kx)} + \frac{B(ik)}{\rho(\omega^2)} e^{i(\omega t + kx)}]$$

$$= \frac{-iA}{\rho c\omega} e^{i(\omega t - kx)} + \frac{iB}{\rho\omega c} e^{i(\omega t + kx)}$$

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$$c^2\left[\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta y^2} + \frac{\delta^2 P}{\delta z^2}\right] = \frac{\delta^2 P}{\delta t^2}$$

$$P(x,t) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)} \ni k = \frac{\omega}{c}$$

$$\left(-\frac{\delta P}{\delta x} = \rho\frac{\delta^2\xi}{\delta t^2}\right) \Leftarrow \text{NEWTON'S LAW}$$

$$\frac{\delta P}{\delta x} = -[ikAe^{i(\omega t - kx)} + ikBe^{i(\omega t + kx)}] = \rho\frac{\delta^2\xi}{\delta t^2}$$

$$\Rightarrow \xi(x,t) = \frac{-iA}{\rho\omega c} e^{i(\omega t - kx)} + \frac{iB}{\rho\omega c} e^{i(\omega t + kx)}$$

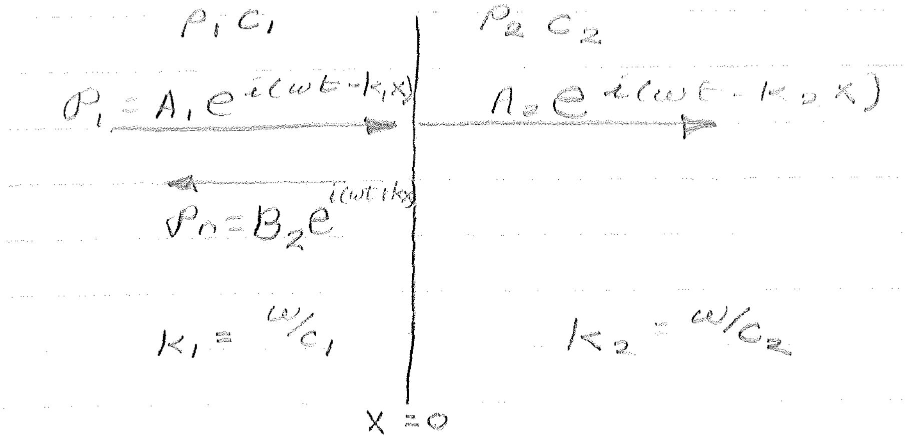
$$U = \text{VELOCITY} = \frac{\delta\xi}{\delta t} = \frac{A}{\rho c} e^{i(\omega t - kx)} - \frac{B}{\rho c} e^{i(\omega t + kx)}$$

SPECIFIC ACOUSTIC IMPEDENCE $\triangleq P/U$

$$\frac{P_x}{U_x} = \frac{Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}}{\frac{A}{\rho c} e^{i(\omega t - kx)} - \frac{B}{\rho c} e^{i(\omega t + kx)}}$$

$$= \rho c\, \frac{Ae^{-ikx} + Be^{ikx}}{Ae^{-ikx} - Be^{ikx}}$$

$\rho c \triangleq$ CHARACTERISTIC IMPEDENCE $(c = \sqrt{B/\rho})$



$$P_{LEFT} = A_1 e^{i(\omega t - k_1 x)} + B_1 e^{i(\omega t + k_1 x)}$$

$$P_{RIGHT} = A_2 e^{i(\omega t - k_2 x)}$$

$$V_L = \frac{A_1}{\rho_1 c_1} e^{i(\omega t\ k_1 x)} - \frac{B}{\rho c} e^{i(\omega t + k_1 x)}$$

$$U_R = \frac{A_2}{\rho_2 c_2} e^{i(\omega t - k_2 x)}$$

BOUNDRY CONDITIONS:

$$P_L \big|_{x=0} = P_R \big|_{x=0} \Rightarrow A_1 + B_1 = A_2$$

$$U_L \big|_{x=0} = U_R \big|_{x=0} \Rightarrow \frac{A_1 - B_1}{\rho_1 c_1} = \frac{A_2}{\rho_2 c_2}$$

$$\therefore \frac{A_1 + B_1}{A_1 - B_1} = \frac{\rho_2 c_2}{\rho_1 c_1}$$

$$\Downarrow$$

$$\frac{B_1}{A_1} = \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1}$$

SIMILARLY; $\frac{A_2}{A_1} = 2\rho_2 c_2 / (\rho_1 c_1 + \rho_2 c_2)$

THEN:

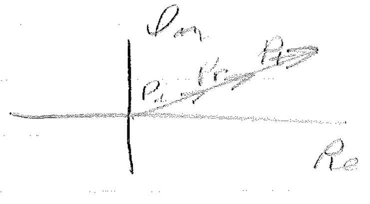
$$P_i = A_1 e^{i(\omega t - kx)}$$

$$P_r = \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1} e^{i(\omega t + kx)}$$

$$P_t = \frac{2\rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} A_1 e^{i(\omega t - kx)}$$

PHASES @ BOUNDRY

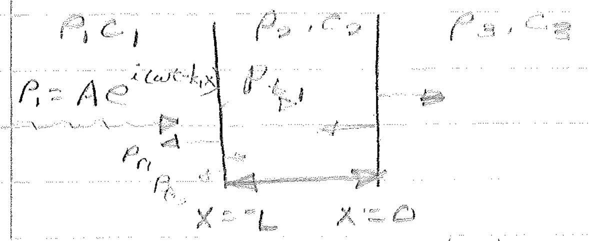
$$P_i \big|_{x=0} = A_1 e^{i\omega t}$$



$$P_r \big|_{x=0} = \underbrace{\frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1}}_{} A e^{i\omega t}$$

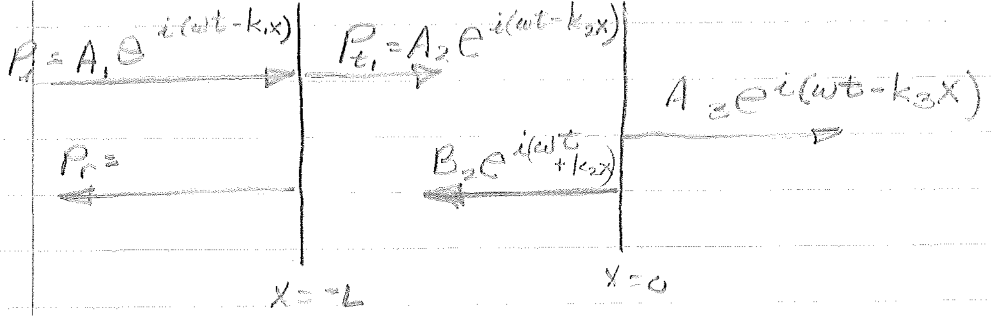
→ SIGN DETERMINES 0 OR 180° PHASE SHIFT

$$P_t \big|_{x=0} = \frac{2\rho_2 c_2}{\rho_1 c_1 + \rho_2 c^2} A_1 e^{i\omega t}$$



$$P_{r_1} = a_1 e^{i(\omega t + k_1 x)}$$
$$P_{t_2} = a_2 e^{i(\omega t + k_1 x)}$$
$$\vdots$$

SUMMING AFTER TRANSIENTS



$$P|_{x=0} \Rightarrow A_2 + B_2 = A_3$$
$$U|_{x=0} \Rightarrow \frac{A_2}{\rho_2 c_2} - \frac{B_2}{\rho_2 c_2} = \frac{A_3}{\rho_3 c_3}$$
$$\Rightarrow \frac{B_2}{A_2} = \frac{\frac{\rho_3 c_3}{\rho_2 c_2} - 1}{\frac{\rho_3 c_3}{\rho_2 c_2} + 1} = \frac{r_{23} - 1}{r_{23} + 1}$$

$\rho_1 c_1$

$$\frac{A_3}{A_2} = \frac{2\rho_3 c_3}{\rho_3 c_3 + \rho_2 c_2}$$

BOUNDRY CONDITIONS @ $x = -L$

$$P \Rightarrow A_1 e^{ik_1 L} + B e^{-ikL} = A_2 e^{ik_2 L} + B_2 e^{-ik_2 L}$$

$$U|_{x=-L} \Rightarrow \frac{A_1}{\rho_1 c_1} e^{ik_1 L} - \frac{B_1}{\rho_1 c_1} e^{-ik_1 L} = \frac{A_2}{\rho_2 c_2} e^{ik_2 L} - \frac{B}{\rho_2 c_2} e^{-ik_2 L}$$
$$\Rightarrow A_1 e^{ik_1 L} - B_1 e^{-ik_1 L} = r_{12} [A_2 e^{ik_2 L} - B_2 e^{-ik_2 L}]$$

$$\Rightarrow \frac{A_1 e^{ik_1 L} + B_1 e^{ikL}}{A_2 e^{ikL} - B_1 e^{ikL}} = r_{12} \frac{e^{ik_1 L} + \frac{B_2}{A_1} e^{-ik_2 L}}{e^{ik_2 L} - \frac{B_2}{A_2} e^{-ik_2 L}}$$

$$= r_{12} \frac{e^{ik_2 L} + \frac{r_{23} - 1}{r_{23} + 1} e^{-ik_2 L}}{e^{ik_2 L} - \frac{r_{23} - 1}{r_{23} + 1} e^{ik_2 L}}$$

$$= r_{12} \frac{2r_{23} \cos k_2 L + 2i \sin k_2 L}{2ir_{23} \sin k_2 L + 2 \cos k_2 L}$$

$$\frac{B_1 e^{-ik_1L}}{A_1 e^{ik_1L}} = \frac{r_{12}\dfrac{r_{23}\cos k_2L + i\sin k_2L}{r_{23}\, i\sin k_2L + \cos k_2L} - 1}{r_{12}\dfrac{r_{23}\cos k_2L + i\sin k_2L}{r_{23} i\sin k_2L + \cos k_2L} + 1}$$

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ALGEBRA FROM ABOVE

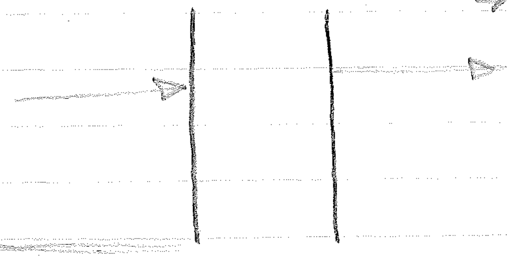
$$\frac{B_1 e^{ik_1L}}{A_1 e^{ik_1L}} = \frac{(r_{12}r_{23}-1)\cos k_2L + i(r_{12}-r_{23})\sin k_2L}{(r_{12}r_{23}+1)\cos k_2L + i(r_{12}+r_{23})\sin k_2L}$$

$$\frac{B_1}{A_1} = \left[\frac{(r_{13}-1)\cos k_2L + i(r_{12}-r_{23})\sin k_2L}{(r_{13}+1)\cos k_2L + i(r_{12}+r_{23}\sin k_2L}\right]e^{2ik_1L}$$

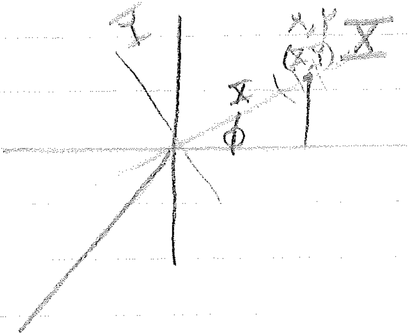
CHOOSE $k_2L = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$

AND $r_{12} = r_{23} \Rightarrow \rho_2C_2 = \sqrt{(\rho_1c_1)\rho_3c_3)}$

$\Rightarrow B_1 = 0$



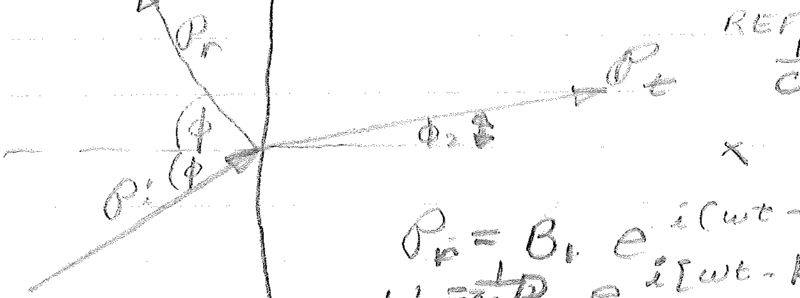
ON LENS



$$\rho = A\, e^{i(\omega t - kx)}$$
$$\rho = A\, e^{i(\omega t - k(x\cos\phi + Y\sin\phi)}$$

x $\dfrac{\delta\rho}{\delta X} = \rho \dfrac{\delta^2\Sigma}{\delta t^2} = \rho \dfrac{\delta U}{\delta X}$

$$U = \frac{A}{\rho_1c_1} e^{i(\omega t - k_1(x\cos\phi + Y\sin\phi)}$$



REFRACTION LAW

$$\frac{1}{c_1}\sin\phi = \frac{1}{c_2}\sin\phi$$

x

$$\rho_r = B_1\, e^{i(\omega t - k_1(x\cos\phi + Y\sin\phi))}$$
$$\qquad\qquad i[\omega t - k_1(-x\cos\phi + Y\sin\phi)]$$

$$U_r = \frac{1}{\rho_1c_1}B_1\, e$$

$$P_t = A_2\, e^{i[\omega t - k(x\cos\phi_2 + Y\sin\phi_2)]}$$

$$U_t = \frac{A_2}{\rho_2c_2} e^{i(\omega t - k(x\cos\phi_2 + Y\sin\phi_2)]}$$

BOUNDRY CONDITIONS

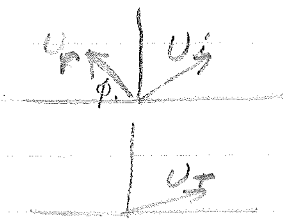
$$P_L|_{x=0} = P_r|_{x=0}$$

$$\Rightarrow A_1 e^{i[\omega t - k y \sin\phi]} + B_1 e^{i(\omega t - k_1 \sin\phi]} = A_2 e^{i\omega t - k_2 y \sin\phi_2}$$

BUT $K_1 \sin\phi_1 = K_2 \sin\phi_2$ $(\omega_1 = \omega_2)$

$$\Rightarrow A_1 + B_1 = A_2$$

COMPONENT NORMAL TO BOUNDRY$|_{x=0}$ = COMP OF U ⊥ TO BOUNDRY$|_{x=0}$



$$\Rightarrow [u_1 \cos\phi_1 - u_r \cos\phi_2]|_{x=0} = u_t \cos\phi_1|_{x=0}$$

$$\left[\frac{A_1}{\rho_1 c_1} - \frac{B_1}{\rho_1 c_1}\right] \cos\phi_1 = \frac{A_2}{\rho_2 c_2} \cos\phi_2$$

$$\Rightarrow \frac{A_1 - B}{A + B} = \frac{\rho_2 c_2}{\rho_1 c_1} \frac{\cos\phi_2}{\cos\phi_1}$$

$$\frac{B}{A} = \frac{\frac{\rho_2 c_2}{\rho_1 c_1} \frac{\cos\phi_2}{\cos\phi_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} \frac{\cos\phi_2}{\cos\phi_1} + 1}$$

IF $\frac{\rho_2 c_2}{\rho_1 c_1} \frac{\cos\phi_2}{\cos\phi_1} = 1 \Rightarrow \cos\phi_1 = \frac{\rho_2 c_2}{\rho_1 c_1} \cos\phi_2$

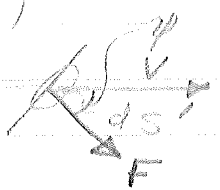
FROM REFRACTION $\Rightarrow \sin\phi_1 = \sqrt{\frac{\rho_1 \left(\frac{c_1}{c_2}\right)^2 - \rho_2^2}{\rho_1^2 - \rho_2^2}}$

ENERGY

$$P = A e^{i(\omega t - kx)}$$

$$dF = P\,ds$$

$$\frac{dE}{\delta t} = P\,ds\,v \qquad (\;Fv)$$



$$\frac{\delta E}{\delta t} = P\,ds'\,v\cos\psi$$

$$\left(\frac{\delta E}{\delta t}\right)_{AVE} = \frac{1}{\tau}\int_0^{\tau} P\,ds\,v\,dt$$

FOR TILT $\left(\frac{\delta E}{\delta t}\right) = \frac{1}{\tau}\int_0^{\tau} P\,ds\,v\cos\psi\,dt$

$$\left(\frac{\delta E}{\delta t}\right)_{AV} = \frac{1}{\tau}\int_0^{\tau} A\cos(\omega t - kx + \alpha)\frac{A}{\rho c}\cos(\omega t - kx + \alpha)\,dt\,ds$$

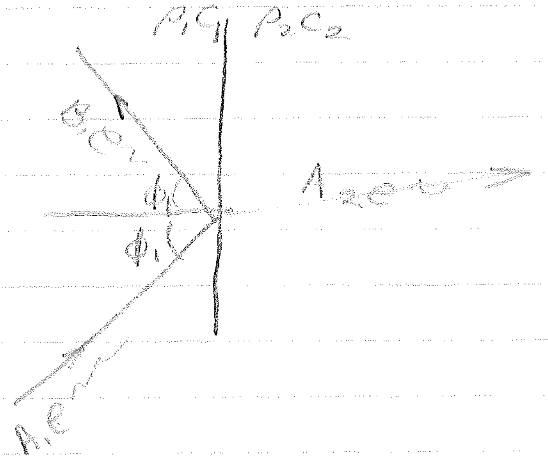
$$= \frac{A^2 ds}{\rho c \tau}\int_0^{\tau}\cos^2(\omega t - kx + \alpha)\,dt$$

$$= \frac{A^2}{2\rho c}\,ds$$

RATE

I = INTENSITY = UNIT AREA @ WHICH ENERGY CROSSES A SMALL SURFACE ⊥ TO DIR. OF PROP

$$I = \frac{A^2}{2\rho c}$$



SO ENERGY BOUNDRY CONDITIONS:

$$\frac{|A_1|^2}{2\rho_1 c_1} ds \cos\phi_1 - \frac{|B_1|^2}{2\rho_1 c_1} ds \cos\phi_1 = \frac{|A_2|^2}{2\rho_2 c_2} ds \cos\phi_2$$

$$\Rightarrow \frac{|A_2|^2}{|A_1|^2} = \frac{\rho_2 c_2}{\rho_1 c_1}\left[1 - \frac{|B_1|^2}{|A_1|^2}\right]\frac{\cos\phi_2}{\cos\phi_1}$$

RECALL $\frac{|B_1|}{|A_1|} = \frac{r_{12} - 1}{r_{12} + 1}$

(GO FORWARD 5 PGS TO 10-30-72)

10-31-72

DUE TUES.

IN BOOK 7.4, 7.8, 7.13, 7.20

ALSO

DOES: $P(r,\theta,t) = \frac{A_1}{r} e^{i(\omega t - kr)} \left[\frac{2J_1(k_a \sin\theta)}{k_a \sin\theta}\right]$

SATISFY THE WAVE EQUATION?

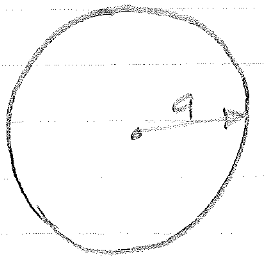
$$P(r,t) = \frac{A}{r} e^{i(\omega t - kx)}$$

$$-\frac{\delta P}{\delta r} = \rho \frac{\delta U}{\delta t}$$

$$\Rightarrow z = \frac{P}{U} = \frac{\rho c kr(kr+i)}{1+(kr)^2} = \frac{\rho c kr}{\sqrt{1+(kr)^2}} e^{i\alpha} \Rightarrow \tan\alpha = \frac{1}{kr}$$

$$U = \frac{A}{rz} e^{i(\omega t - kr)}$$

PULSATING SPHERE



$$U_s = U_0 e^{i\omega t}$$

$$U_0 e^{i\omega t} = \frac{A}{a z_{r=a}} e^{i\omega t} e^{ika}$$

$$\Rightarrow A = a U_0 e^{ika} z_{r=a}$$

$$e^{ika} z_{r=a} = (\cos ka + i \sin ka)\left[\frac{\rho c ka(ka+i)}{1+(ka)^2}\right]$$

GET RID OF HIGHER TERMS

$$= \rho c\left[1 - \frac{ka^2}{2} + i(ka)\right]$$
$$\left[ka(ka+i)\left(1+(ka)^2+\dots\right)\right]$$
$$= \rho c\left[ka - \frac{(ka)^3}{2} + i(ka)^2\right]$$
$$\left[ka - (ka)^3 + i - i(ka)^2\right]$$
$$= \rho c i ka$$

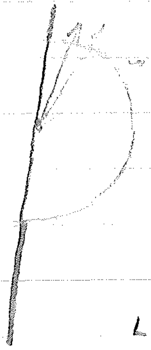
$$\Rightarrow A \approx i\rho c ka^2 U_0 \quad \text{WHEN} \quad ka \ll 1$$

AND $P(r,t) = \frac{A}{r} e^{i(\omega t - kr)}$

$$\approx \frac{i\rho c ka^2 U_0}{r} e^{i(\omega t - kx)}$$

$$= \frac{i\rho c k Q_s}{4\pi r} e^{i(\omega t - kr)}$$

$Q_s = 2\pi a^2 U_0$ = SOURCE STRENGTH



$$p = \frac{i\rho c k Q_H}{2\pi r} e^{i(\omega t - kr)}$$

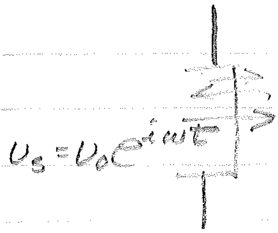
LET $\lambda = \frac{2\pi}{k} \gg$ SOURCE DIMENSIONS

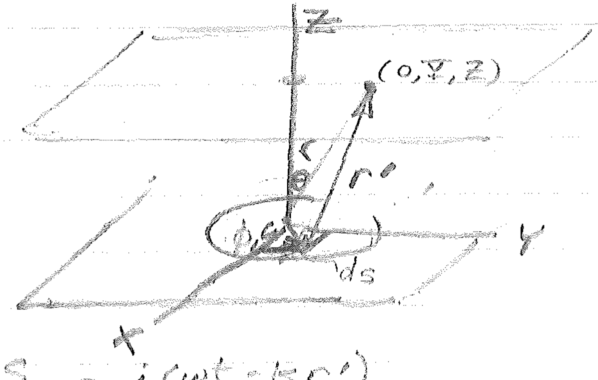
$r \quad\quad \gg$ " "

$Q_s = Q_H$

~~TH~~ AT THESE CONDITIONS, ANY SOURCE WILL ACT LIKE THE HEMISPHERICAL SOURCE

$$dp(r,t) = \frac{i\rho c k\, dQ}{4\pi r} e^{i(\omega t - kx)}$$



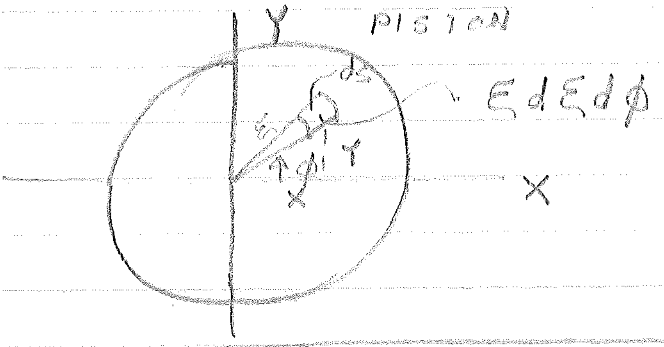


$$dp = \frac{i\rho c k U_0\, ds}{2\pi r'} e^{i(\omega t - kr')}$$

$$p = \int_S \frac{i\rho c k U_0}{2\pi r'} e^{i(\omega t - kr')} ds$$

FOR $r \approx r'$

$$p = \frac{i\rho c k U}{2\pi r} \cdot e^{i(\omega t - kr)} \pi a^2 \quad (0^{TH} \text{ ORDER})$$



$$r' = \sqrt{(0-X)^2 + (\Upsilon - Y)^2 + (Z-0)^2}$$
$$= \sqrt{X^2 + Y^2 + \Upsilon^2 + Z^2 - 2Y\Upsilon}$$
$$X = \xi\cos\phi \quad ; \quad Y = \xi\sin\phi$$
$$\Upsilon^2 + Z^2 = r^2 \quad ; \quad \Upsilon = r\sin\phi$$
$$\Rightarrow r' = \sqrt{\xi^2 + r^2 - 2r\xi\sin\theta\sin\phi}$$
$$= r\left[1 + \left\{\left(\frac{\xi}{r}\right)^2 - 2\frac{\xi}{r}\sin\theta\sin\theta\right\}\right]^{1/2}$$

NOW $\frac{a}{r} \ll 1 \Rightarrow \frac{\xi}{r} \ll 1$

$$= r\left[1 + \frac{1}{2}\left(\frac{\xi}{r}\right)^2 - \frac{\xi}{r}\sin\theta\sin\phi\right]$$
$$\simeq r\left[1 - \frac{\xi}{r}\sin\theta\sin\phi\right]$$

$$\Rightarrow p = \frac{i\rho c k U_0}{2\pi r} e^{i\omega t}\int_0^{2\pi}\int_0^{a} e^{-ik\left[r - \frac{\xi}{r}\sin\theta\sin\phi\right]}\xi d\xi d\phi$$
$$= \frac{i\rho c k U_0}{2\pi r} e^{i(\omega t - kr)}\int_0^{2\pi} d\phi\int_0^a e^{ik\xi\sin\theta\sin\phi}\xi d\xi$$
$$\beta = ik\sin\theta\sin\phi$$
$$\int_0^a e^{\beta\xi}\xi d\xi$$
$$U = \xi \qquad dV = e^{\beta\xi}d\xi$$
$$dU = d\xi \qquad V = \frac{1}{\beta}e^{\beta\xi}$$
$$\int_0^a e^{\beta\xi}\xi d\xi = \frac{\xi e^{\beta\xi}}{\beta}\Big|_0^a - \frac{1}{\beta}\int_0^a e^{\beta\xi}d\xi$$
$$= \frac{a e^{\beta a}}{\beta} - \frac{1}{\beta^2}\left[e^{\beta a} - 1\right]$$
$$= \left[\frac{a}{\beta} - \frac{1}{\beta^2}\right]e^{\beta a} - \frac{1}{\beta^2}$$
$$\int d\phi\int\xi e^{\beta\xi}d\xi = \int d\phi\left(\left[\frac{a}{\beta} - \frac{1}{\beta^2}\right]\left[1 + \beta a + \frac{(\beta a)^2}{2!} + \frac{(\beta a)^3}{3!} + \dots\right] + \frac{1}{\beta^2}\right)$$
$$= \int d\phi\left[\frac{a}{\beta} + a^2 + \frac{a^3\beta}{2!} + \frac{a^4\beta^2}{3!} + \frac{a^5\beta^4}{4!} + \right.$$
$$\left. - \frac{1}{\beta^2} - \frac{a}{\beta} - \frac{a^2}{2!} + \frac{a^3\beta}{3!} + \dots + \frac{1}{\beta^2}\right]$$

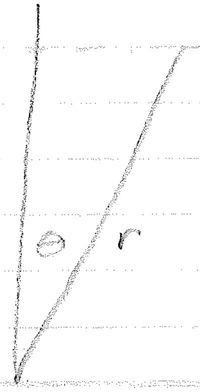
$$= a^2 \int_0^{2\pi} d\phi \left[\frac{1}{2} + \frac{\beta a}{3} + \frac{(\beta a)^2}{8} + \frac{(\beta a)^3}{30} + \dots \right]$$

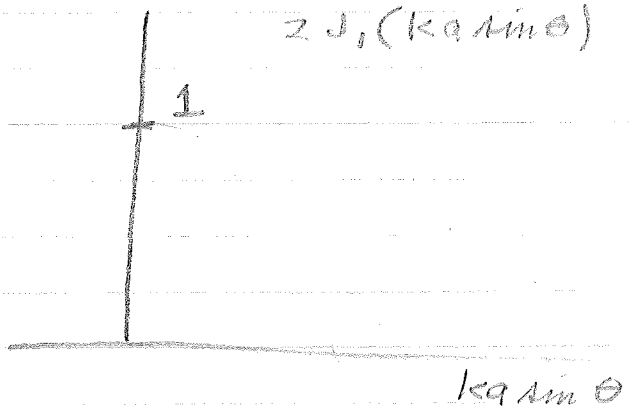
$$= a^2 \int_0^{2\pi} \left[\frac{1}{2} + \frac{ika \sin\theta \sin\phi}{3} + \frac{(ika \sin\theta \sin\phi)^2}{8} + \frac{(ika \sin\theta \sin\phi)^3}{20} + \frac{(\quad)^4}{144} + \dots \right] d\phi$$

$$\pi a^2 \left[1 - \frac{(ka \sin\theta)^2}{8} + \frac{(ka \sin\theta)^4}{(\quad)} + \dots \right]$$

$$= \pi a^2 \left[\frac{2J_1(ka \sin\theta)}{ka \sin\theta} \right]$$

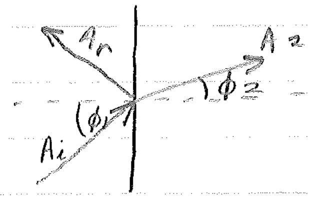
$$\Rightarrow p = \frac{i\rho c k u_0}{2\pi r} e^{i(\omega t - kr)} \pi a^2 \left[\frac{2J_1(ka \sin\theta)}{ka \sin\theta} \right]$$





(GO UP TO 11-1-72)

10-30-72



FOR ENERGY BALANCE:

$$\frac{|A_1|^2}{2\rho_1 c_1}\cos\phi_1 - \frac{|B_1|^2}{2\rho_1 c_1}\cos\phi_1 = \frac{|A_2|}{2\rho_2 c_2}\cos\phi_2$$

SPHERICAL WAVES

$$\mathcal{P} = -B_a\left[\frac{\delta\xi}{\delta x} + \frac{\delta\eta}{\delta y} + \frac{\delta\zeta}{\delta z}\right] = -B_a \operatorname{div}\vec{s}$$

$$-\operatorname{grad}\mathcal{P} = \rho\frac{\delta^2 \vec{s}}{\delta t^2} = \rho\frac{\delta V}{\delta t}$$

$$\vec{s} = \begin{cases} s_r \\ s_\theta \\ s_\phi \end{cases} \qquad \vec{V} = \begin{cases} v \\ v \\ w \end{cases} \qquad \begin{aligned} -\operatorname{div}\overrightarrow{\operatorname{grad}}\mathcal{P} &= \rho\frac{d^2}{\delta t^2}\operatorname{div}\vec{s} \\ -\operatorname{div}\overrightarrow{\operatorname{grad}}\mathcal{P} &= \rho\frac{d^2}{\delta t^2}\left(-\frac{\mathcal{P}}{B_a}\right) \end{aligned}$$

$$\Rightarrow c^2 \operatorname{div}\overrightarrow{\operatorname{grad}}\mathcal{P} = \frac{d^2\mathcal{P}}{\delta t^2}$$

$$c^2\left[\frac{\delta^2\mathcal{P}}{\delta r^2} + \frac{2}{r}\frac{\delta\mathcal{P}}{\delta r} + \frac{1}{r^2}\frac{\delta^2\mathcal{P}}{\delta\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{\delta\mathcal{P}}{\delta\phi} + \frac{1}{r^2\sin^2\theta}\frac{\delta^2\mathcal{P}}{\delta\phi^2}\right] = \frac{\delta^2\mathcal{P}}{\delta t^2}$$

LET $\mathcal{P}(r,t) = R(r)H(t)$

$$\Rightarrow \frac{c^2}{R}\left[\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right] = \frac{1}{H}\frac{d^2H}{dt^2} = -\omega^2$$

$$\Rightarrow H = a_2\cos\omega t + b_2\sin\omega t$$

AND $\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} = -k^2R$ $\Rightarrow k = \omega/c$

$$\frac{d^2}{dr^2}(rR) = -k^2RV$$

$$\Rightarrow rR = a_1\cos kr + b_1\sin kr$$

$$\therefore R = \frac{a_1}{r}\cos kr + \frac{b_1}{r}\sin kr$$

$$\Rightarrow P(r,t) = \frac{A}{r}\cos(kr+\gamma)\cos(\omega t+\varphi)$$

$$-\text{grad}\, P = \rho \frac{\delta^2 \vec{s}}{\delta t^2} = \rho \frac{\delta \vec{V}}{\delta t}$$

$$\left\{ \begin{array}{l} -\frac{\delta P}{\delta r} = \rho \frac{\delta^2 s}{\delta t^2} = \rho \frac{\delta V}{\delta t} \\ -\frac{1}{r}\frac{\delta P}{\delta \phi} = \rho \frac{d^2 s_\phi}{\delta t^2} = \rho \frac{\delta V}{\delta V} \\ -\frac{dP}{r \sin\phi\, \delta\phi} = \rho \frac{d^2 s_\phi}{\delta t^2} = \rho \frac{\delta w}{\delta t} \end{array} \right\}$$

$$P(r,t) = \frac{A}{r} e^{i(\omega t - kr)}$$

$$\rho \frac{dV}{\delta t} = -\left[\frac{A}{r^2} e^{i(\omega t - kV} + \frac{A}{r}(-ik) e^{i(\omega t - kV)}\right]$$

$$U = -\left[-\frac{A}{r^2 i\omega\rho} e^{i(\omega t - kr)} + \frac{A(-ik)}{r(i\omega)\rho} e^{i(\omega t - kr)}\right]$$

$$= \frac{1}{\rho c}\left[\frac{1}{ikr} + 1\right]\frac{A}{r} e^{i(\omega t - kr)}$$

DEFINE THE SPECIFIC ACOUSTIC IMPEDENCE

$$z = \frac{P}{U} = \frac{\rho c\, kr(kr+i)}{1+(kr)^2}$$

$$= \frac{\rho c k r e^{i\alpha}}{\sqrt{1+(kr)^2}} \ni \alpha = \tan^{-1}\left(\frac{1}{kr}\right)$$

$$I = \frac{1}{\tau}\int_0^\tau P\, U\, dt$$

$$U_{REAL} = \frac{A\sqrt{1+(kr)^2}}{\rho c\, k r^2} \cos(\omega t - kr + \Psi + \alpha)$$

$$\Rightarrow I = \frac{1}{\tau}\int_0^\tau \frac{A^2\sqrt{1+(kr)^2}}{(\rho c k r) r^2} \cos(\omega t - kr - \varphi) \cos(\omega t - kr - \Psi - \alpha)\, d$$

$$= \frac{A^2\sqrt{1+(kr)^2}}{(\rho c k r) r^2} \cdot \frac{\cos\alpha}{2} = \frac{A^2\sqrt{1+(kr)^2}}{\rho c k r (r^2)} \cdot \frac{kr}{2\sqrt{1+(kr)^2}}$$

$$= \frac{A^2}{2\rho c r^2}$$

PULSATING SPHERICAL SOURCE

$$V_s = U_0 e^{i\omega t}$$

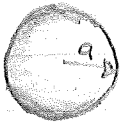
$$P = \frac{A}{r} e^{i(\omega t - kr)}$$

$$V = \frac{A}{r z} e^{i(\omega t - kr)} \Rightarrow z = \frac{\rho c\, kr(kr+i)}{[1+(kr)^2]}$$

$$V_s = v\big|_{r=a}$$

$$V_0 e^{i\omega t} = \frac{A}{a z\big|_{r=a}} e^{-ika} e^{i\omega t}$$

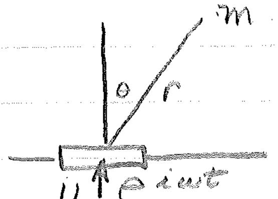
$$A = V_0 a e^{ika} z_m\big|_{r=a}$$



$$e^{ika} z\big|_{r=a} = (\cos ka + i \sin ka)\left(\frac{\rho c\, ka(ka+i)}{1+(ka)^2}\right]$$

GO BACK TO 10-31-72

11-1-72



$$P = \frac{i\rho c k U_0 a^2}{2r}\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]e^{i(wt-kr}$$

$J_1 = 0$ @ 3.83, 7.02, 10.17

SO WHEN $\sin\theta\, ka = 3.83, 7.02, 10.13, \ldots$

$$\sin\theta = \frac{3.83}{ka}\ \frac{7.02}{ka}\ \frac{10.13}{ka}$$

$$= \frac{3.83\lambda}{2\pi a},\ \frac{7.02\lambda}{2\pi a}$$

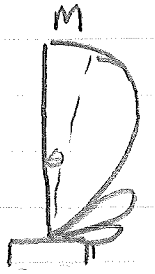
(LIKE LIGHT GIVING BESSEL FUNCTION)

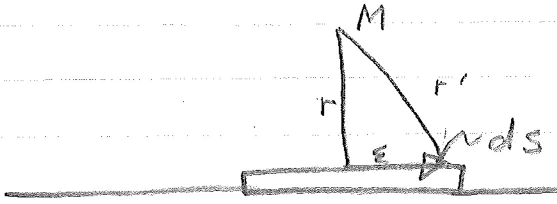
AGAIN $\frac{2\pi}{\lambda} a \sin\theta = 3.83, 7.02, 10.17, \ldots$

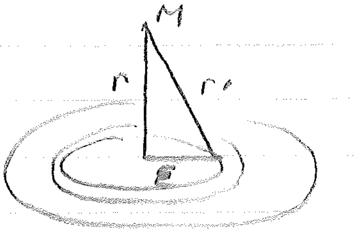
LET $a = 10\lambda$

$\Rightarrow 6.28 \sin\theta = 3.83, \ldots$ $\Rightarrow$ NO SOLUTION FOR θ

$\therefore \frac{2\pi}{\lambda} a$ DETERMINS NUMBER OF 0'S ON θ SWEEP.







$$dP = \frac{i\rho c k U ds}{2\pi r'} e^{i(wt - kr')}$$

$$P = \int \frac{i\rho c k U_0}{2\pi\sqrt{\varepsilon^2 + r^2}} 2\pi \varepsilon d\varepsilon e^{i(wt - k\sqrt{\varepsilon^2 + r^2}}$$

$$= i\rho c k U_0 e^{iwt} \int \frac{\varepsilon d\varepsilon\, e^{-ik\sqrt{\varepsilon^2 + r^2}}}{\sqrt{\varepsilon^2 + r^2}}$$

LET $V = -ik\sqrt{\varepsilon^2 + r^2} \Rightarrow dv = -ik \frac{\varepsilon d\varepsilon}{\sqrt{\varepsilon^2 + r^2}}$

$$\Rightarrow P = -\rho c U_0 e^{iwt} \int e^v dv = \rho c U_0 e^{iwt}\left[e^{-ik(a^2 + r^2)} - e^{-ikr}\right]$$

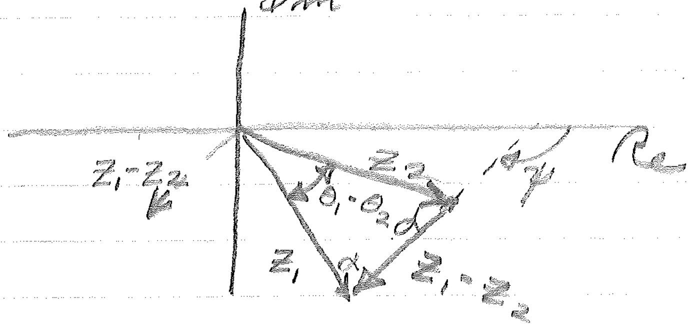
AGAIN

$$P = -\rho c U_0 e^{i\omega t}\left[e^{-ik\sqrt{a^2+r^2}} - e^{-ikr}\right]$$

LET $z_1 = e^{-ik\sqrt{r^2+a^2}}$; $z_2 = e^{-ikr}$

$\theta_1 = k\sqrt{r^2+a^2}$; $\theta_2 = k.r$ $\Big\}$

$$\Rightarrow P = -\rho c U_0 e^{i\omega t}\left[z_1 - z_2\right]$$



$z_1 - z_2 = |z_1 - z_2| e^{i\psi} \ni |z_1 - z_2| = \sqrt{1+1+2\cos(\theta_1-\theta_2)}$

$= \sqrt{2}\sqrt{1-\cos(\theta_1-\theta_2)}$

$= \sqrt{2}\sin\frac{\theta_1-\theta_2}{2}$

$2\alpha + (\theta_1-\theta_2) = 180 \Rightarrow \alpha = 90 - \frac{\theta_1-\theta_2}{2}$

$\psi = \alpha + \theta_1 = 90 - \frac{\theta_1}{2} + \frac{\theta_2}{2} + \theta_1 = 90 + \frac{\theta_1+\theta_2}{2}$

$\Rightarrow |z_1 - z_2| = 2\sin\frac{\theta_1-\theta_2}{2}\, e^{-i\pi/2}\, e^{-i(\theta_1+\theta_2)/2}$

$= -2i\sin\frac{\theta_1-\theta_2}{2}\, e^{-i(\theta_1+\theta_2)/2}$

$\therefore P = i\rho c U_0 e^{i\omega t}\sin\frac{\theta_1-\theta_2}{2}\, e^{-i\frac{\theta_1+\theta_2}{2}}$

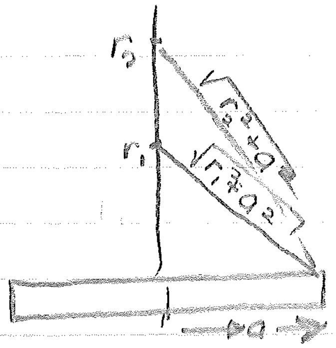
$= -\rho c U_0 \sin\frac{\theta_1-\theta_2}{2}\left[\sin\left(\omega t - \frac{\theta_1+\theta_2}{2}\right)\right]$

$= -\rho c U_0 \sin\left(\frac{k\sqrt{r^2+a^2} - kr}{2}\right)\sin\left(\omega t - \frac{k\sqrt{r^2+a^2}+kr}{2}\right)$

$= -\rho c U_0 e^{i\omega t}\left[z_1 - z_2\right] = 0$ FOR $\theta_1 = \theta_2 + 2\pi n$

OR $k\sqrt{r^2+a^2} - kr = n2\pi$

$\sqrt{r^2+a^2} - r = n\lambda = \lambda, 2\lambda, 3\lambda, \ldots$



$$\sqrt{r_{MOST\ DISTANT}^2 + a^2} - r_{MOST} = \lambda$$

$$r_M^2 + a^2 = \lambda^2 + 2 r_a \lambda + r^2$$

$$r_{MOST\ DIST} = \frac{a^2 - \lambda^2}{2\lambda}$$

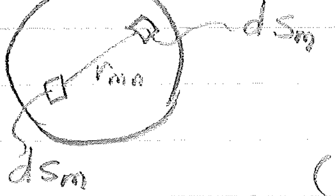
$$\ddot{Y}_0 = \frac{F_0 e^{i\omega t}}{Z_m}$$

BEFORE STRING

$$\dot{Y}_0 = \frac{F_0 e^{i\omega t}}{Z_m + Z_{iN}}$$

AFTER

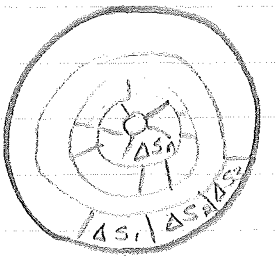
x



$$dP_{nm} = \frac{i\rho c k v_0}{2\pi r} dS_m$$

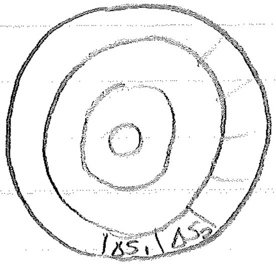
$$df_{nm} = \frac{i\rho c k v_0}{2\pi r} dS_m dS_n$$

$$(df_{nm})_x = \frac{-i\rho c k v_0}{2\pi r} dS_m dS_n$$
$$= (df_{mn})_x$$



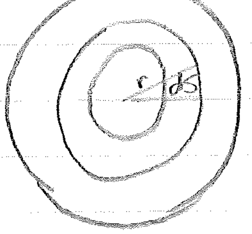
$$\begin{aligned} F_x = &\left[\Delta P_{12} + \Delta P_{13} + \Delta P_{14} + \dots + \Delta P_{1N}\right]\Delta S_1 \\ - &\left[\Delta P_{21} + \Delta P_{23} + \Delta P_{24} + \dots + \Delta P_{2n}\right]\Delta S_2 \\ - &\left[\Delta P_{31} + \Delta P_{32} + \Delta P_{34} + \dots + \Delta P_{3N}\right]\Delta S_3 \\ &\vdots \end{aligned}$$

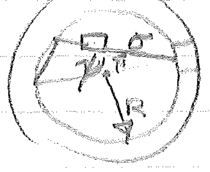
11-2-72



$$F_x = -[\Delta P_{12} + \Delta P_{13} + \Delta P_{14} + \Delta P_{15} + \ldots + \Delta P_{1N}]\Delta S$$
$$- [\cancel{\Delta P_{21}} + \Delta P_{23} + \Delta P_{24} + \Delta P_{25} + \ldots]\Delta S_2$$
$$- [\cancel{\Delta P_{31}} + \cancel{\Delta P_{32}} + \Delta P_{34} + \ldots]\Delta S_3$$
$$- [\cancel{\Delta P_{41}} + \Delta P_{42} + \ldots]\Delta S_4$$

$$F_x = -2 \sum_{m=1}^{N} \Delta S_m \sum_{n=m+1}^{N} \Delta P_{mn}$$





$$F_x = -2\int_0^a 2\pi R\,dR \int_{-\pi/2}^{\pi/2}\int_0^{2R\cos\psi} \frac{i\rho c k U e^{i(\omega t - k\sigma)}}{2\pi\sigma}\,\sigma\, d\sigma\, d\psi$$
$$= 2\rho c U_0 e^{i\omega t}\int_0^a R\,dR \int_{-\pi/2}^{\pi/2}\int_0^{2R\cos\psi} -ik e^{-ik\sigma} d\phi$$

$$= 2\rho c U_0 e^{i\omega t}\int_0^a R\,dR \int_{-\pi/2}^{\pi/2} [e^{-ik2R\cos\psi} - 1]\,d\psi$$

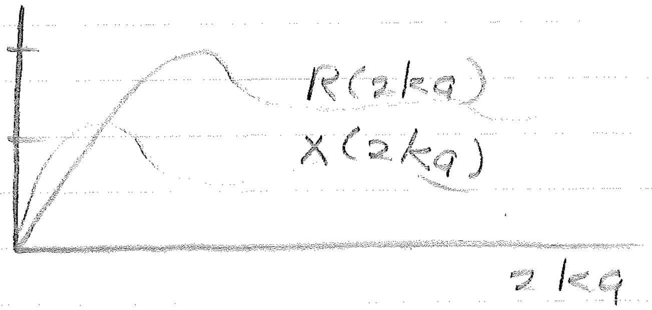
$$= +2\rho c U_0 e^{i\omega t}\int_0^a R\,dR \int_{-\pi/2}^{\pi/2} [1 - ik2R\cos\psi + \frac{(-ik2R\cos\psi)^2}{2!} + \frac{(ik2R\cos\psi)^3}{3!} + \frac{(-ik2R\cos\psi)^4}{4!} + \ldots - 1]\,d\psi$$

$$= 2\rho c U_0 e^{i\omega t}\int_0^a R\,dR \int_{-\pi/2}^{\pi/2} [(-\frac{(k2R)^2}{2}\cos^2\psi + \ldots) - i(k2R\cos\psi) + \ldots]\,d\psi$$
$$= 2\rho c U_0 e^{i\omega t}\int_0^a R\,dR \left[\frac{-k^2 4R^2}{2}\left(\frac{\psi}{2} - \frac{1}{4}\sin 2\psi\right)\Big|_{-\pi/2}^{\pi/2} - ik2\pi\sin\psi\Big|_{-\pi/2}^{\pi/2}\right]$$
$$= 2\rho c U_0 e^{i\omega t}\int R\,dR\,[-k^2 2R^2\frac{\pi}{2} + \ldots - ik4R + \ldots]$$
$$= -2\rho c U_0 e^{i\omega t}\int R\,dR\,[+k^2 R^2\frac{\pi}{2} + \ldots + i(k4R + \ldots)]$$

$$= -2\rho c e^{i\omega t}[k^2\pi\frac{a^4}{4} + \ldots + i(\frac{k4a^3}{3}$$

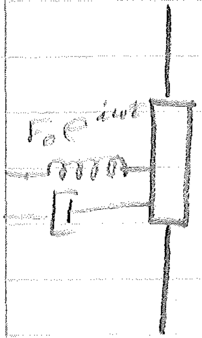
$$F_x = \rho c U_0 e^{i\omega t} \pi a^2 \left\{ \left[\frac{(ka)^2}{2} + \dots \right] + i \left[\frac{8}{3\pi} ka + \dots \right] \right\}$$
$$= \rho c U_0 e^{i\omega t} \pi a^2 \left\{ R_1(2ka) + i X_1(2ka) \right\}$$

$\ni$ $R_1(2ka)$ = PISTON RESISTANCE FUNCTION
$X(2ka)$ = " REACTANCE "



RADIATION IMPEDANCE OF THE PISTON

$$Z_R = \frac{\text{X COMPONENT OF FORCE EXERTED BY PISTON ON THE WAVE}}{\text{VELOCITY OF PISTON}}$$
$$= \rho c \pi a^2 \left[R_1(2ka) + i X(2ka) \right]$$



$$m\ddot{x} + R\dot{x} + kx = F_0 e^{i\omega t}$$

$$\Rightarrow \dot{x} = \frac{F_0 e^{i\omega t}}{Z_m} \ni Z_m = R + i\left(\omega m - \frac{k}{\omega}\right)$$

WITH WAVE:

$$m\ddot{x} + R\dot{x} + kx = F_0 e^{i\omega t} - Z_r U_0 e^{i\omega t}$$

$$\Rightarrow \dot{x} = \frac{F_0 e^{i\omega t} - Z_r U_0 e^{i\omega t}}{Z_m}$$

$$\Rightarrow Z_m = \frac{F_0 e^{i\omega t}}{\dot{x}} - Z_R \ni \dot{x} = \frac{F_0 e^{i\omega t}}{Z_m + Z_R}$$

ENERGY (ON TEST?)

$$dW = \vec{F} \cdot d\vec{s} \quad ; \quad m\ddot{x} + R\dot{x} + kx = F_0 e^{i\omega t}$$

$$\frac{dW}{dt} = \vec{F} \cdot d\vec{V}$$

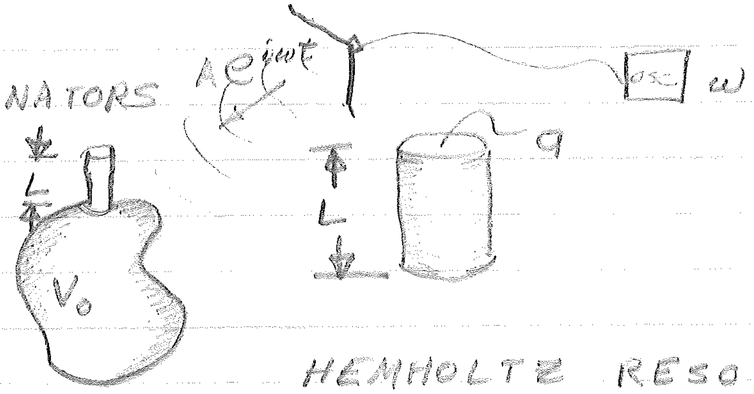
ave rt of energy dissipation $= \frac{1}{T}\int_0^T R(\dot{x})^2 dt$

FOR $\dot{x} = \frac{F_0 e^{i\omega t}}{Z_m + Z_r}$

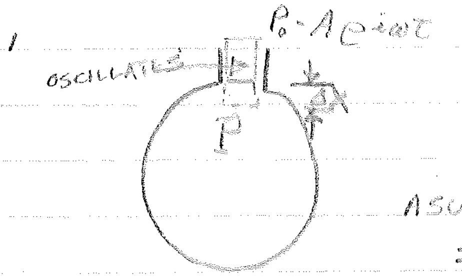
$$E_{AV} = \frac{1}{T}\int_0^T (\rho c \pi a^2) R_1(2ka) U_0^2 \cos^2(\omega t + \phi) dt$$

$$= \rho c \pi a^2 R_1(2ka) U_0/2$$

RESONATORS



HEMHOLTZ RESONATOR



$$P(\pi a^2) - (P_0 - Ae^{i\omega t})\pi a^2 = \rho\pi a^2 l \frac{\delta^2 x}{\delta t^2}$$

ASUME PV^{δ} = CONSTANT

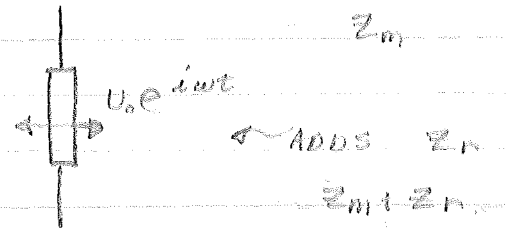
$$\Rightarrow dP = \frac{\delta P_0}{V_0} dV = P - P_0$$

$$dV = \pi a^2 x$$

$$\Rightarrow P = P_0 - \frac{\delta P_0}{V_0} \pi a^2 x$$

$$\Rightarrow P_0 \pi a^2 - \frac{\delta P_0}{V_0}(\pi a^2)^2 x - P_0 \pi a^2 + A\pi a^2 e^{i\omega t} = \rho\pi a^2 l \frac{d^2 x}{dt^2}$$

11-6-72



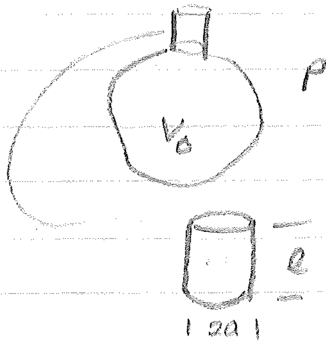
$$Z_r = \rho c \pi a^2 [R_1(2ka) + i X(2ka)]$$

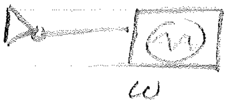
$$\simeq \rho c \pi a^2 \left[\frac{(ka)^2}{2} + i \frac{8}{3\pi} ka\right] \qquad \ni k = \frac{\omega}{c}$$

$$= \left[\rho c \pi a^2 \frac{(ka)^2}{2}\right] + i \frac{8}{3\pi} \frac{\omega}{c} \rho c \pi a^3$$

$$= \rho c \pi a^2 \frac{(ka)^2}{2} + i\omega \left[\frac{8}{3\pi} \rho \pi a^3\right]$$

NOW $z = R + i(\omega M - K/\omega) \Rightarrow \Delta m = \frac{8}{3\pi} \rho \pi a^3$



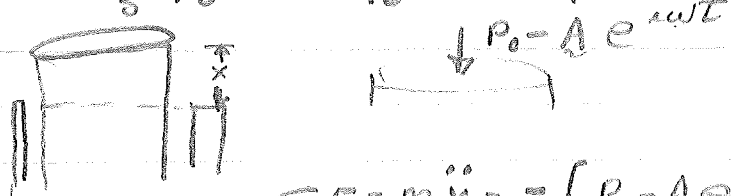


m = MASS OF GAS IN THE NECK
$= \rho \pi a^2 \ell$

IDEAL GAS $\Rightarrow PV^{\gamma}$ = CONSTANT

$$\Rightarrow dP = \frac{-\gamma P_0}{V_0} dV$$

$$P_g - P_0 = \frac{-\gamma P_0}{V_0} \pi a^2 X$$



$$\Sigma F = m\ddot{x} = -[P_0 - Ae^{i\omega t}]\pi a^2 + \left[P_0 - \frac{\gamma P_0}{V_0} \pi a^2 x\right]\pi a^2 - R\dot{x}$$

$$\Rightarrow m\ddot{x} + R\dot{x} - \frac{\gamma P_0}{V_0}(\pi a^2)^2 x = A\pi a^2 e^{i\omega t}$$

LET $K = \dfrac{\gamma P_0 (\pi a^2)^2}{V_0}$

$F_0 = A \pi a^2 e^{i\omega t}$

$$\Rightarrow m\ddot{x} + R\dot{x} + Kx = F_0 e^{i\omega t}$$

$$\Rightarrow \dot{x} = \frac{F_0 e^{i\omega t}}{R + i(\omega m - K/\omega)}$$

$$Re(\dot{x}) = F_0 \cos(\omega t - \theta + \alpha) / \sqrt{R^2 + (\omega m - K/\omega)^2}$$

$$x = \frac{-i F_0 e^{i\omega t}}{\omega[R + i(\omega m - K/\omega)]}$$

$$Re(x) = \frac{F_0 \sin(\omega t - \theta + \alpha)}{\omega \sqrt{R^2 + (\omega m - K/\omega)^2}}$$

$$P_g - P_0 = \frac{-\gamma P_0}{V_0} \pi a^2 x$$

$$= \frac{-\gamma P_0}{V_0} \pi a^2 \frac{F_0}{\omega \sqrt{R^2 + (\omega m - K/\omega)^2}} \sin(\omega t - \theta \alpha)$$

RESONANCE: $\omega_{RES} = \sqrt{K/m}$ FOR $\dot{x}$

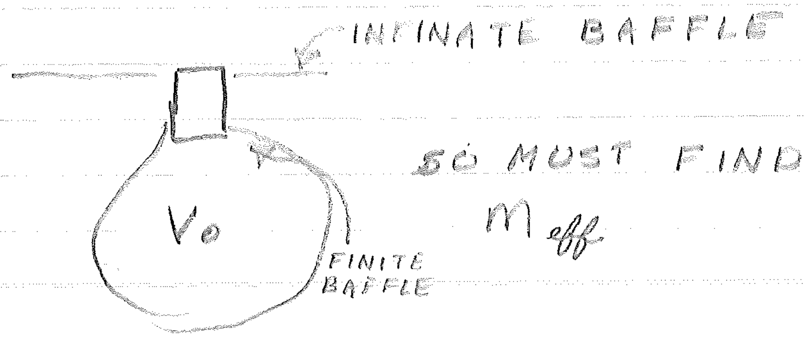
$\omega_{RES} = \sqrt{\dfrac{K}{m} - \dfrac{R^2}{2m}}$? FOR X

BUT, ITS GOOD ENUFF TO USE:

$$\omega_{RES} = \sqrt{\frac{K}{m}} = \sqrt{\frac{\gamma P_0 (\pi a^2)^2}{V_0 m}} = \sqrt{\frac{\gamma P_0 (\pi a^2)^2}{V_0 \rho (\pi a^2) l}}$$

$$= \sqrt{\frac{c^2 (\pi a^2)}{V_0 l}} = c\sqrt{\frac{\pi a^2}{V_0 l}}$$

BUT IT'S TO BIG. (ω_{RES} (EXPERIMENT) $<$ ω_{RES})



SO MUST FIND M_{eff} R_{eff} (EFFECTIVE)

AGAIN

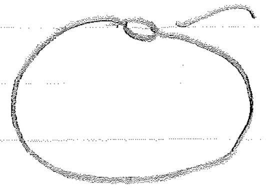
$$\omega_{RES} = \sqrt{\frac{K}{m}} = \sqrt{\frac{\gamma P_0 (\pi a^2)^2}{V_0 [\rho \pi a^2 \gamma + \frac{16}{3\pi} \rho \pi a^2 a}}$$

$$= \sqrt{\frac{\gamma P_0 (\pi a^2)^2}{V_0 \rho \pi a^2 [\ell + \frac{16}{3\pi} a]}}$$

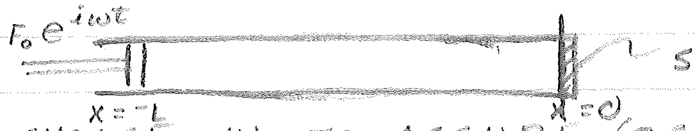
$$= \sqrt{\frac{\gamma P_0 (\pi a^2)^2}{V_0 \rho \pi a^2 \ell_i'}}$$

ℓ_i' = EFFECTIVE LENGTH

NO NECK!



$\ell' = \frac{16}{3\pi} a$



CHOOSE ω TO ASSURE (0,0) MODE ($<\omega_{cut}$)
GIVING PLANE WAVES

$$m\ddot{x}_p + R\dot{x}_p + KX = F_0 e^{i\omega t}$$

FOR NO REFLECTED WAVES

$$\dot{x}_p = \frac{F_0 e^{i\omega t}}{Z_m} \ni Z_m = R + i(\omega m - K/\omega)$$

BUT WAVES GIVE MORE MASS AND R

$$\Rightarrow m'\ddot{X}_p + R'\dot{X}_p + KX = F_0 e^{i\omega t}$$

WAVE IN PIPE:

$$P = a e^{i(\omega t - kx)} + b e^{i(\omega t + kx)}$$
$$U = \frac{a}{\rho c} e^{i(\omega t - kx)} - \frac{b}{\rho c} e^{i(\omega t + kx)}$$

$$U(x=0)=0$$
$$\Rightarrow \frac{a}{\rho c} e^{i\omega t} - \frac{b}{\rho c} e^{i\omega t} = 0$$
$$\Rightarrow a = b$$
$$\Rightarrow P = a e^{i\omega t}[e^{ikx} + e^{ikx}]$$
$$= 2a \cos kx\, e^{i\omega t} = A \cos kx\, e^{i\omega t}$$
$$U = \frac{-1}{\rho c} A i \sin kx\, e^{i\omega t}$$

THEN: $m'\ddot{X} + R'x + KX = F_0 e^{i\omega t} - A \cos kx \Big|_{x=-L} S e^{i\omega t}$

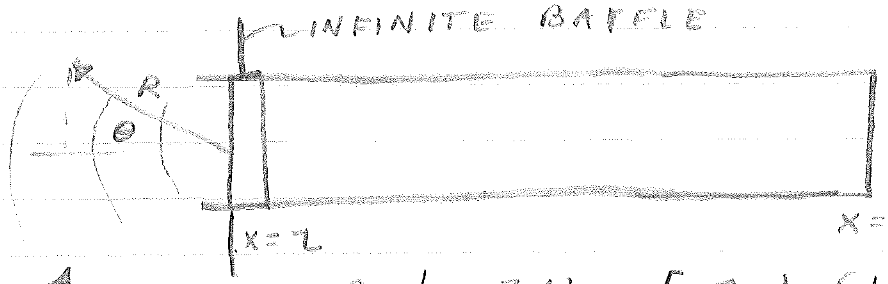
YIELDING:

$$\dot{x} = \frac{[F_0 - AS\cos kL]e^{i\omega t}}{Z_m + Z_r}$$

$$Z_m + Z_r = \frac{F_0 e^{i\omega t}}{\dot{x}} - \frac{(A S \cos kL) e^{i\omega t}}{\rho c A i \sin kL \, e^{i\omega t}}$$

$$Z_m + Z_r - i\rho c S \cot kL = \frac{F_0 e^{i\omega t}}{\dot{x}}$$

$$\dot{x} = \frac{F_0 e^{i\omega t}}{Z_m + Z_r - i\rho c S \cot kL}$$



RECALL $$P = \frac{\rho c k a^2 U_0}{2r} \left[\frac{2 J_1(ka \sin\theta)}{ka \sin\theta} \right]$$ (ARS)

IF $Z_m >> Z_r - i\rho c S \cot kL$

$$\dot{x} = \frac{F_0 e^{i\omega t}}{\rho c \pi a^2 [R_1(2ka) + i X(2ka)] - i\rho c S \cot kL}$$

$$= \frac{F_0 e^{i\omega t}}{\{[\rho c S R_1(2ka)]^2 + [\rho c S [X(2ka) - \cot kL]]^2\}^{1/2}}$$

$\dot{x}_{MAX}$ FOR $\cot kL = \underbrace{X_1(2ka)}_{\text{CONSTANT}}$

$Z_m \not\gg Z_r - i\rho c S \cot kL$, SO DRIVE PIPE WITH A SPEAKER

11-7-72

$$P = a_1 e^{i(\omega t - kx)} + b_1 e^{i(\omega t + kx)}$$

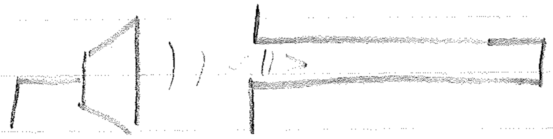
$$U = \frac{a_1}{\rho c} e^{i(\omega t - kx)} - \frac{b_1}{\rho c} e^{i(\omega t + kx}$$

$$U|_{x=0} = 0 \Rightarrow \underline{a} = \underline{b}$$

$$P = A e^{i\omega t} \cos kx$$

$$U = \frac{iA}{\rho c} e^{i\omega t} \sin kx$$

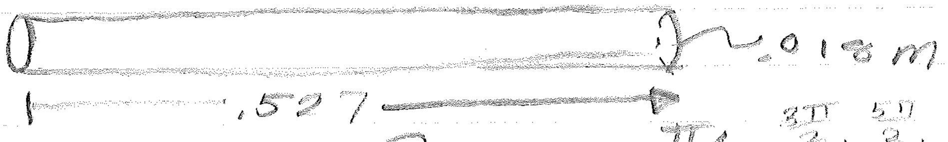
$$\dot{X}_p = \frac{F_0 e^{i\omega t}}{Z_m + Z_r - i\rho c S \cot kL}$$



$$\dot{X}_p = \frac{F_0 e^{i\omega t}}{\rho c S [R_1(2ka) + iX_1(2ka)] - i\rho c S \cot kL}$$

$$= \frac{F_0/\rho c S \quad \cos(\omega t + \alpha)}{\sqrt{R_1^2(2ka) + [X_1(2ka) - \cot kL]^2}}$$

$$\underset{\text{LITTLE}}{\uparrow} \qquad\qquad \underset{\text{LITTLE}}{\uparrow}$$



$\dot{X}_p$ RESONATES @ $\sim kL = \pi/2, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$ BUT A BIT BEFORE

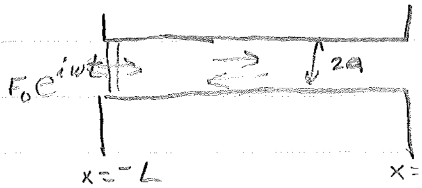
HARMONICS: 1, 3, 5, 7, . . .

$$\text{NOW} \quad \dot{X}_p = \frac{F_0 e^{i\omega t} - P|_{x=L} S}{Z_m + Z_r} = \frac{+iAe^{i\omega t} \sin kL}{\rho c}$$

$$\dot{X}_p = \frac{F_0 e^{i\omega t} - A e^{i\omega t} \cos kL}{Z_m + Z_r}$$

$$= \frac{iAe^{i\omega t} \sin kL}{\rho c}$$

11-8-72



$$p = \underset{\sim}{A} e^{i(\omega t - kx)} + \underset{\sim}{B} e^{i(\omega t + kx)}$$

$$\mu = \frac{A}{\rho c} e^{i(\omega t - kx)} - \frac{B}{\rho c} e^{i(\omega t + kx)}$$

$$\mu\Big|_{x=0} = \left[\frac{A}{\rho c} - \frac{B}{\rho c}\right] e^{i\omega t} = U_0 e^{i\omega t}$$

$$-F_x = \rho c \pi a^2 U_0 e^{i\omega t}[R_1 + i X_1] \quad @_{x=0}$$

$$\Rightarrow p = \frac{\rho c S U_0 e^{i\omega t}[R_1 + i X_1]}{S}$$

$p_{x=0}$ FROM VIB = $p_{x=0}$ EXTERNAL:

$$A e^{i\omega t} + B e^{i\omega t} = \frac{\rho c S U_0 e^{i\omega t}[R_1 + i X)}{s} \Rightarrow Z_r = \rho c S[R_1 + i$$

$$\Rightarrow A + B = Z_r U_0 / S$$

$$\Rightarrow \left[\frac{A_1}{\rho c} - \frac{B_1}{\rho c}\right] e^{i\omega t} = U_0 e^{i\omega t}$$

$$\Rightarrow \frac{A_1 + B_1}{A_1 - B_1} = \frac{Z_r}{\rho c S} \quad \text{FROM} \Rightarrow \frac{B}{A} = \frac{Z_r - \rho c S}{Z_r + \rho c S}$$

NOW THE PISTON: $\dot{X}_p = \dfrac{F_0 e^{i\omega t} - p|_{s=-L} S}{Z_r + Z_m}$

AND

$$p = A e^{i\omega t}\left[e^{-ikx} + \frac{Z_r - \rho c S}{Z_r + \rho c S} e^{ikx}\right]$$

$$= A e^{i\omega t}\left[\frac{2 Z_r \cos kx - i 2\rho c S \sin kx}{Z_r + \rho c S}\right]$$

THEN: $\dot{X}_p = \dfrac{F_0 e^{i\omega t} - A S e^{i\omega t}\left[\dfrac{2 Z_r \cos kL + i 2\rho c S \sin kL}{Z_r + \rho c S}\right]}{Z_r + Z_m}$

$$U = \frac{1}{\rho c} A e^{i\omega t}\left[e^{-iks} - \frac{Z_r - \rho c S}{Z_r + \rho c S} e^{ikx}\right]$$

$$= \frac{1}{\rho c} A e^{i\omega t}\left[\frac{-i 2 Z_r \sin kx + 2\rho c S \cos kx}{Z_r + \rho c S}\right]$$

$$\Rightarrow \dot{x}_0 = \frac{1}{\rho c} A e^{i\omega t}\left[\frac{2iZ_r \sin kL + 2\rho c S \cos kL}{Z_r + \rho c S}\right]$$

PUTTING TWO $\dot{x}_0$ EXPRESSIONS TOGETHER:

$$\rho c F_0[Z_n+\rho c S] = A\left[\{Z_n+Z_m)(2iZ_r\sin kL + 2\rho c S\cos kL) + \rho c S[2Z_n\cos kL + 2i\rho c S\sin kL)\right]$$

$$\Rightarrow A = \frac{\rho c F_0[Z_r+\rho c S]}{(Z_r+Z_m)(2iZ_r\sin kL+2\rho c S\cos(kL)+\rho c S[2Z_r\cos kL + 2i\rho c S\sin kL}$$

$$= \frac{\rho c F_0[Z_r+\rho c S]}{2(Z_r^2+(\rho c S)^2) i \sin kL + 4Z_r\rho c S\cos kL}$$

↑
FOR $Z_m >> Z_r$

RECALL:

$$P = A e^{i\omega t}\left[\frac{2Z_r\cos kx - i2\rho c S\sin kx}{Z_r+\rho c S}\right]$$

$$\Rightarrow P\Big|_{x=0} = \frac{2Z_r\rho c F_0 \quad e^{i\omega t}}{2[Z_r^2+(\rho c S)^2]\, i\sin kL + 4Z_r\rho c S\cos kL}$$

RECALL $Z_r = \rho c S[R_1 + iX]$

$$\Rightarrow P\Big|_{x=0} = \frac{\rho c S[R_1+iX_1]\rho c F_0 \quad e^{i\omega t}}{(\rho c S)^2[(R+iX_1)^2+1]\, i kL + 2(\rho c S)^2(R_1+iX)\cos kL}$$

$$= \frac{F_0/S\ [R_1+iX_1]\, e^{i\omega t}}{[(R_1+iX_1)^2+1]\, i\sin kL + 2(R_1+iX_1)\cos kL}$$

$$P\Big|_{x=0} = \frac{F_0/S\ \sqrt{R_1^2+X_1^2}\, e^{i\alpha}\, e^{i\omega t}}{[(R_1^2-X_1^2+2iX_1R_1)+1]\, i\sin kL + 2R_1\cos kL + 2iX_1\cos kL}$$

$$= \frac{\frac{F_0}{S}\sqrt{R_1^2+X_1^2} \qquad \cos(\omega t+\alpha-\theta)}{\sqrt{[-2R_1X_1\sin kL + 2R_1\cos kL]^2 + [(1+R_1^2-X_1^2)\sin kL + 2X_1\cos kL]^2}}$$

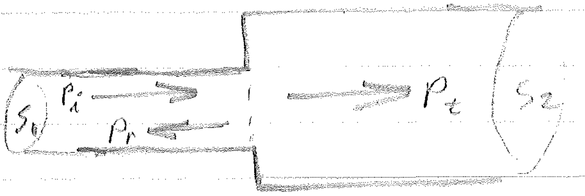
LOOK @ THE DENOMINATOR

$$\frac{F_0/S\ \sqrt{R_1^2+X_1^2} \qquad \cos \omega t}{\sqrt{(2R_1)^2[X_1\sin kL + \cos kL]^2 + (\underbrace{\sin kL}_{\text{DOMINATE TERM}} + 2X_1\cos(kL))]^2}}$$

CAUSE X_1 & R_1 ARE Teenie

RESONANCE @ ABOUT $kL = n\pi$

WAVES IN PIPES



$$P_L = A e^{i(\omega t - kx)} + B e^{i(\omega t + kx)}$$
$$P_R = A_2 e^{i(\omega t - kx)}$$

AND: $U_L = \frac{A}{\rho c} e^{i(\omega t - kx)} - \frac{B}{\rho c} e^{i(\omega t + kx)}$

$$U_R = \frac{A_2}{\rho c} e^{i(\omega t - kx)}$$

BOUNDARY CONDITIONS;

1) $P_L|_{x=0} = P_R|_{x=0}$

2) $U_L S_1 = U_R S_2$ $\Rightarrow$ US = VOLUME VELOCITY

YIELDING:

$$A_1 + B_1 = A_2$$

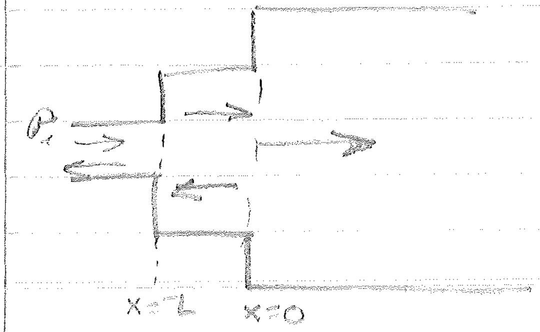
AND $\left(\frac{A_1}{\rho c} - \frac{B_1}{\rho c}\right) S_1 = \frac{A_2}{\rho c} S_2$

$$\frac{A_1 + B_1}{A_1 - B_1} = \frac{S_1}{S_2} \Rightarrow \frac{B}{A_1} = \frac{S_1/S_2 - 1}{S_1/S_2 + 1}$$

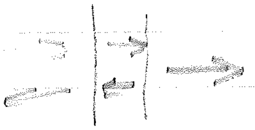
ENEGY CONSIDERATIONS:

$$\frac{|A|^2}{2\rho c} S_1 = \frac{|B_1|}{2\rho c} S_1 + \frac{|A_2|^2}{2\rho c} S_2$$

(MAY USE EITHER)



ANALAGOUS TO:



11-9-72

ACOUSTIC IMPEDANCE

$$Z^* = P/uS \quad ; \quad uS = \text{VOLUME VELOCITY}$$

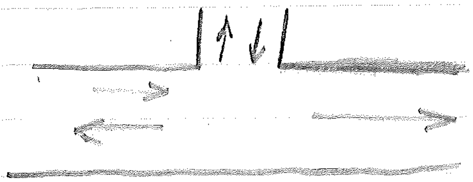
$$P = A e^{i(wt-kx)} + B e^{i(wt+kk)}$$

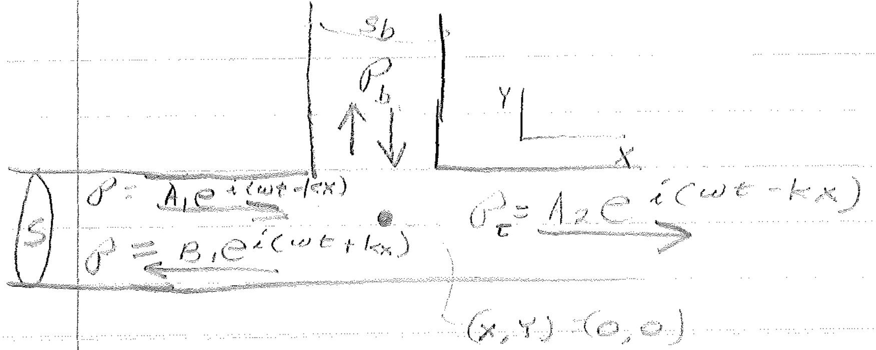
$$U = \frac{A}{\rho c} e^{i(wt-kx)} - \frac{B}{\rho c} e^{i(wt+kx)}$$

$$\Rightarrow Z^* = \frac{\rho c}{S} \frac{A e^{-ikx} + B e^{ikz}}{A e^{ikx} - B e^{ikx}}$$

FOR A SINGLE WAVE (ie B=0)

$$Z^* = \frac{\rho c}{S}$$





$$P_L\big|_{x=0} = P_b\big|_{y=0} = P_t\big|_{x=0}$$

$$U_1\big|_{x=0} S = U_b\big|_{y=0} S_b + U_t\big|_{x=0} S$$

$$\left[\frac{U_1 S}{P} = \frac{U_b S_b}{P_b} + \frac{U_t S}{P_t}\right]_{x=y=0}$$

$$\frac{1}{P/SU} = \frac{1}{\frac{P}{U_b S_b}} + \frac{1}{P_t/U_t S}$$

$$\Rightarrow \frac{1}{Z^*} = \frac{1}{Z_b^*} + \frac{1}{Z_t^*}$$

$$\Rightarrow Z^* = \frac{Z_b^* Z_t^*}{Z_b^* + Z_t^*}$$

$$\frac{\rho c}{S}\,\frac{A_1+B_1}{A_1-B_1} = \frac{Z_b^*\,\rho c/S}{Z_b^* + \rho c/S}$$

$$\frac{A_1+B_1}{A_1-B_1} = \frac{Z_b}{Z_b + \rho c/S}$$

$$\Rightarrow \frac{B_1}{A_1} = \frac{Z_b - (Z_b + \frac{\rho c}{S})}{Z_b + (Z_b + \frac{\rho c}{S})}$$

$$= \frac{-\rho c/S}{2 Z_b + \frac{\rho c}{S}}$$

$$= \frac{-\rho c/2S}{Z_b^* + \rho c/2S}$$

DEFINE: INTENSITY RATIO $\alpha_T = \dfrac{\frac{|A_2|^2}{2\rho c} S}{\frac{|A_1|^2}{2\rho c} S} = \left|\dfrac{A_2}{A_1}\right|^2$

$$\tilde{A}_1 + \tilde{B}_1 = \tilde{A}_2$$

$$1 + \frac{B_1}{A_1} = \frac{A_2}{A_1}$$

$$\Rightarrow \frac{A_2}{A_1} = 1 - \frac{\rho c/5s}{z_b + \frac{\rho c}{2s}} = \frac{z_b}{z_b + \rho c/2s}$$

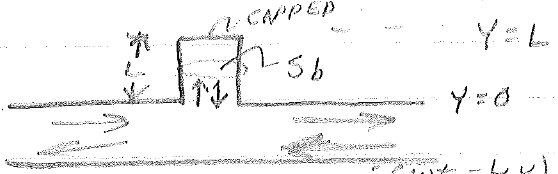
THUS $\alpha = \dfrac{|z_b|^2}{|z_b + \rho c/2s|^2}$

LET $z_b = R_b + i X_b \Rightarrow |z_b|^2 = R_b^2 + X_b^2$

$$z_b + \frac{\rho c}{2s} = R_b + \frac{\rho c}{2s} + i X_b$$

$$\Rightarrow \left|z_b + \frac{\rho c}{2s}\right|^2 = \left(R_b + \frac{\rho c}{2s}\right)^2 + X_b^2$$

$$\therefore \alpha_t = \frac{R_b^2 + X_b^2}{\left(R_b + \frac{\rho c}{2s}\right)^2 + X_b^2}$$



$$P_b = A_b e^{i(\omega t - ky)} + B_b e^{i(\omega t + kx)}$$

$$U_b = \frac{A_b}{\rho c} e^{i(\omega t - ky)} + \frac{B}{\rho c} e^{i(\omega t + kx)}$$

$$U_b\big|_{y=L} = 0 \Rightarrow \frac{A_b}{\rho c} e^{-ikL} + \frac{B_b}{\rho c} e^{ikL}$$

$$\Rightarrow \frac{B_b}{A_b} = e^{-2ikL}$$

FOR ANY POINT: $z_b = \dfrac{\rho c}{S_b} \dfrac{A_b e^{-iky} + B_b e^{iky}}{A_b e^{-iky} - B_b e^{iky}}$

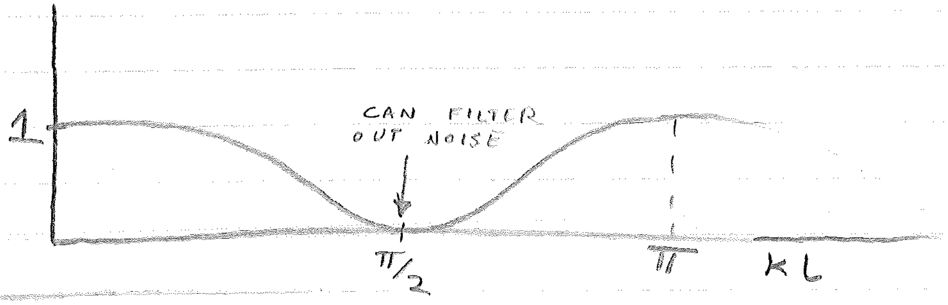
$$z_b\big|_{y=0} = \frac{\rho c}{S_b}\left[\frac{A_b + B_b}{A_b - B_b}\right]$$

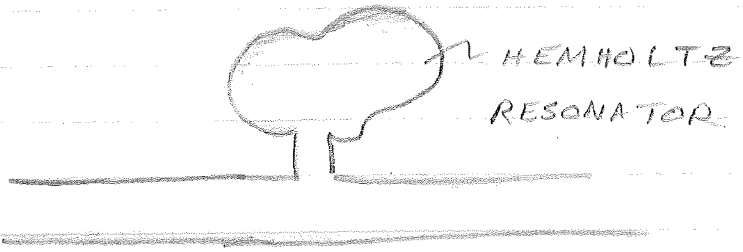
$$= \frac{\rho c}{S_b}\left[\frac{A_b + A_b e^{-2ikL}}{A_b - A_b e^{-2ikL}}\right]$$

$$= \frac{\rho c}{S_b} \frac{e^{ikL} + e^{-ikL}}{e^{ikL} - e^{ikL}}$$

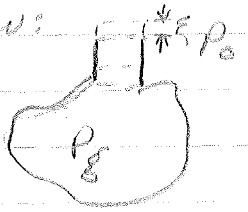
$$= \frac{-i\rho c}{S_b} \cot kL$$

AND $\alpha_t = \dfrac{(\rho c/S_b)^2 \cot^2(kL)}{\left(\frac{\rho c}{2s}\right)^2 + \left(\frac{\rho c}{S_b}\right)^2 \cot^2(kL)} = \dfrac{1}{\left[\frac{S_b}{2s}\right] \tan^2 kL + 1}$





REVIEW:



$$PV^\gamma = \text{CONST} \Rightarrow dP_\gamma = \frac{\gamma P_0}{V_0} dV$$

$$P_\xi - P_0 = \frac{\gamma P_0}{V_0} S \xi$$

$$m\ddot{\xi} = R\dot{\xi} + \left[P_0 - \frac{\gamma P_0}{V_0} S \xi\right] S - \left[P_0 - A e^{i\omega t}\right] S$$

$$\Rightarrow m\ddot{\xi} + R\dot{\xi} + \underbrace{\left(\frac{\gamma P_0}{V_0} S^2\right)}_{K}\xi = SAe^{i\omega t}$$

$$\Rightarrow \dot{\xi} = \frac{SAe^{i\omega t}}{Z_m} \ni Z_m = R + i(\omega m - K/\omega)$$

NEED TO PATCH:

$$R_{EFF} = R + \rho c S R_1(2ka)$$

$$M_{EFF} = m + \rho S l' \ni l' = l + \frac{16}{3\pi} a$$

$$\Rightarrow \dot{\xi} = \frac{SAe^{i\omega t}}{Z_m'}$$

$$\frac{Z_m'}{S^2} = \frac{Ae^{i\omega t}}{\dot{\xi} S}$$

$$Z_b^* = \frac{Z_m'}{S^2} \ni Z_m' = R_{eff} + i(\omega M_{eff} - K/\omega)$$

BUT R_{eff} IS NEGLIGIBLE (NO ENERGY LOSS)

THEN $Z_b^* = \dfrac{Z_m'}{S^2} \approx \dfrac{i(\omega m' - K/\omega)}{S^2}$

11-14-72

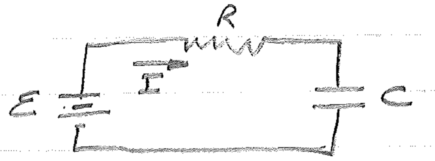
$$\left.\begin{array}{l} P = -B_a \frac{\delta \xi}{\delta x} \\ -\frac{\delta P}{\delta x} = \rho \frac{\delta^2 \xi}{\delta t^2} \end{array}\right\} \Rightarrow c^2 \frac{\delta^2 P}{\delta x^2} = \frac{\delta^2 P}{\delta t^2}$$

$$P = A e^{i(\omega t - kx)} \ni k = \omega/c ,\ c = \sqrt{B_a/\rho}$$

$$P = A_0 e^{-\alpha x} e^{i(\omega t - k'x)} \qquad \ni k' = \frac{\omega}{c'}$$

c' & α ARE FUNCTION OF ω

STOKES SAID: $P = -B_a \frac{\delta \xi}{\delta x} - R \frac{\delta}{\delta t}\left(\frac{\delta \xi}{\delta x}\right)$



$\tau = RC$

$$\tau = \frac{R}{B}$$

GIVING

$$\frac{\delta^2 P}{\delta x^2} = -B_a \frac{\delta}{\delta x} \frac{\delta^2 \xi}{\delta t^2} - R \frac{\delta}{\delta t} \frac{\delta}{\delta x} \frac{\delta^2 \xi}{\delta t^2}$$

$$\frac{\delta^2 P}{\delta x^2} = \quad = -B_a \frac{\delta}{\delta x}\left(-\frac{1}{\rho} \frac{\delta P}{\delta x}\right) - R \frac{\delta}{\delta t} \frac{\delta}{\delta x}\left(-\frac{1}{\rho} \frac{\delta P}{\delta x}\right)$$

$$= \frac{B_a}{\rho} \frac{\delta^2 P}{\delta x^2} + \frac{R}{\rho} \frac{\delta}{\delta t} \frac{\delta^2 P}{\delta x^2}$$

$$\frac{\delta^2 P}{\delta t^2} = c^2 \frac{\delta^2 P}{\delta x^2} + c^2 \tau \frac{\delta}{\delta t} \frac{\delta^2 P}{\delta x^2}$$

IS $P = A_0 e^{-\alpha x} e^{i(\omega t - k'x)}$ A SOLUTION?

$$\frac{\delta P}{\delta t^2} = -\omega^2 P$$

$$\frac{\delta P}{\delta x} = -\alpha P - ik'P$$

$$\frac{\delta^2 P}{\delta x^2} = \alpha P + ik'\alpha P + ik'\alpha P - k'^2 P$$

$$\frac{\delta}{\delta t} \frac{\delta^2 P}{\delta x^2} = i\omega\alpha^2 P - 2\omega k'\alpha P - i\omega k'^2 P$$

$$-\omega^2 = c^2[\alpha^2 + 2ik'\alpha - k'^2] + c^2\tau[i\omega\alpha^2 - 2\omega k'\alpha - i\omega k'^2$$

$$-\frac{\omega^2}{k'^2} = [\alpha^2 + 2ik'\alpha - k'^2] + i\omega t[\alpha^2 + 2ik\alpha - k'^2$$

$$\frac{-\omega^2}{c^2(1 + i\omega\tau)} = \alpha^2 - k'^2 + 2ik\alpha$$

EQUATING REAL AND IMAGINARY PARTS:

$$\left\{\begin{array}{l} \frac{-\omega^2}{c^2(1 + \omega^2\tau^2))} = \alpha^2 - k'^2 \\ \frac{\omega^3 \tau}{c^2[1 + (\omega\tau)^2]} = 2k'\alpha \end{array}\right.$$

$$\frac{\omega^2}{c^2[1+(\omega\tau)^2]} = \alpha^2 - \frac{\omega^6 \tau^4}{c^4[1+(\omega\tau)^2]^2 4\alpha^2}$$

$$\Rightarrow 0 = \alpha^4 + \frac{\omega^2\alpha^2}{c^2[1+(\omega\tau)^2]} - \frac{\omega^6\tau^2}{c^4[1+(\omega\tau)^2]4}$$

$$\alpha^2 = \frac{-\omega^2}{2c^2[1+(\omega\tau)^2]} + \sqrt{\left(\frac{\omega^2}{c^2[1+(\omega\tau)^2]}\right)^2 + \frac{\omega_6\tau^2}{c^4[1+(\omega\tau)^2]^2}}$$

WHICH BOILS DOWN TO:

$$\alpha^2 = \frac{-\omega^2}{2c^2[1+(\omega\tau)^2]} + \frac{1}{2}\frac{\omega^2}{c^2[1+(\omega\tau)^2]}\sqrt{1+(\omega\tau)^2}$$

$$= \frac{\omega^2}{2c^2[1+(\omega\tau)^2]}\left[\sqrt{1+(\omega\tau)^2} - 1\right]$$

$$\alpha = \frac{\omega}{\sqrt{2}\,c\sqrt{1+(\omega\tau)^2}}\left[\sqrt{1+(\omega\tau)^2} - 1\right]^{1/2}$$

THEN:

$$k' = \frac{\omega^3\tau}{\frac{2\omega}{\sqrt{2}c\sqrt{1+(\omega\tau)^2}}\left[\sqrt{1+(\omega\tau)^2} - 1\right] c^2[1+(\omega\tau)^2]} = \frac{\omega}{c'}$$

AND

$$c' = \left[\frac{\sqrt{2}\sqrt{1+(\omega\tau)^2}}{\omega\tau}\left\{\sqrt{1+(\omega\tau)^2} - 1\right\}^{1/2}\right]c$$

LET $\omega\tau << 1$

THEN $\sqrt{1+(\omega\tau^2)} \cong 1 + \frac{1}{2}(\omega\tau)^2$

$$c' = c$$

AND $\alpha = \frac{\omega^2\tau}{2c} \Rightarrow \frac{\alpha}{\omega^2} = \frac{\tau}{2c}$

LET $\omega\tau >> 1$

THEN $\sqrt{1+(\omega\tau)^2} = \omega\tau$

$$c' = \sqrt{2}\sqrt{\omega\tau}\,c$$

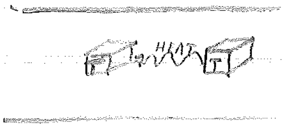
$$\alpha = \frac{1}{c}\sqrt{\omega}/2\tau$$

VISCOUSITY

$F_x = \mu A \frac{\delta u}{\delta r}$

STOKES SHOWED: IF $\omega\tau << 1$; $\alpha = \frac{2}{3}\frac{\mu\omega^2}{\rho c^3}$ BUT IT WAS TOO SMALL

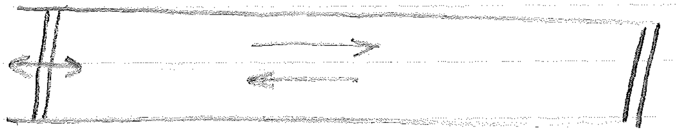
KIRCHOFF SAID



IF $\omega\tau << 1$ $\alpha = \frac{(\gamma - 1)K}{2\rho c^2 C_p}\omega^2$ BUT STILL TO SMALL

BUT, IF YOU ADD THE TWO, YOU GET THE CLASSICAL COEFFICIENT OF ABSORBSION

MEASUREMENT of α



$$P = P_1 + P_2$$

$$P = A_0 e^{-\alpha x} e^{i(\omega t - kx)} + B_0 e^{\alpha x} e^{i(\omega t + kx)}$$

$$-\frac{\delta P}{\delta x} = \rho \frac{\delta u}{\delta t}$$

$$-\alpha P_1 - ikP_1 + \alpha P_2 + ikP_2 = \rho \frac{\delta u}{\delta t}$$

$$+\frac{\alpha}{i\omega\rho}P_1 + \frac{ik}{i\omega\rho}P_1 - \frac{\alpha}{i\omega\rho}P_2 + \frac{ik}{i\omega\rho}P_2 = u$$

$$\left[\frac{\alpha}{i\omega\rho} + \frac{k}{\omega\rho}\right]A_0 e^{-\alpha x} e^{i(\omega t - kx)}$$

$$- \left[\frac{\alpha}{i\omega\rho} + \frac{k}{\omega\rho}\right]B_0 e^{\alpha x} e^{i(\omega t + kx)} = u$$

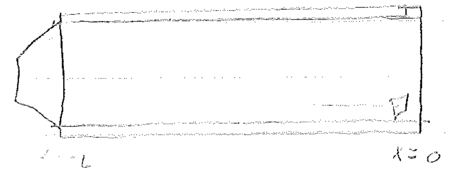
$$u_{x=0} = 0 \Rightarrow A_0 = B_0$$

THEN

$$P = A_0 e^{i\omega t}\left[\{e^{-\alpha x}[\cos kx - i \sin kx) + e^{\alpha x}(\cos kx + i \sin kx)\right]$$

$$= A_0 e^{i\omega t}\left[\cos kx \left[\frac{\cosh \alpha x}{2}\right] + i \sin kx \left[\frac{\sinh \alpha x}{2}\right]\right\}$$

11-15-72



$$\rho = A_0 e^{-\alpha x} e^{i(\omega t - kx)} + B_0 e^{\alpha x} e^{i(\omega t + kx)}$$

$$\mu_{x=0} = 0 \Rightarrow A_0 = B_0$$

$$\rho = A_0 e^{i\omega t}\left[e^{-\alpha}(\cos kx - i \sin kx) + e^{\alpha}(\cos kx + i \sin kx)\right]$$

$$= 2A_0 e^{i\omega t}\left[\underline{\cosh \alpha x \cos kx + i \sin kx \sinh \alpha x}\right]$$

$$\rho_{REAL} = 2|A_0|\sqrt{\cosh^2 \alpha x \cos^2 kx + \sinh^2 \alpha x \sin^2 kx}$$

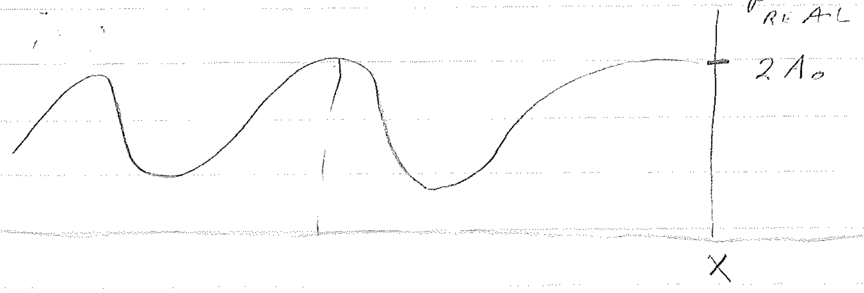
$$\cos(\omega t + \alpha + B)$$

$$\cosh_{\alpha x} = \frac{e^{\alpha x} + e^{-\alpha x}}{2} = \frac{1 + \alpha x - \frac{(\alpha x)^2}{2!} + \dots + 1 - \alpha x + \frac{(\alpha x)^2}{2!}}{2}$$

$$\sinh \alpha x = \frac{1 + \alpha x + \frac{(\alpha x)^2}{2} - \left[1 - \alpha x + \frac{(\alpha x)^2}{2!}\right]}{2}$$

LL T

$$\cosh \alpha x = 1 + \frac{(\alpha x)^2}{2} \; ; \; \sinh \alpha x = \alpha x - \frac{(\alpha x)^3}{3!}.$$

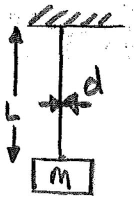


KNOW Def
BOUNDRY CONDITIONS

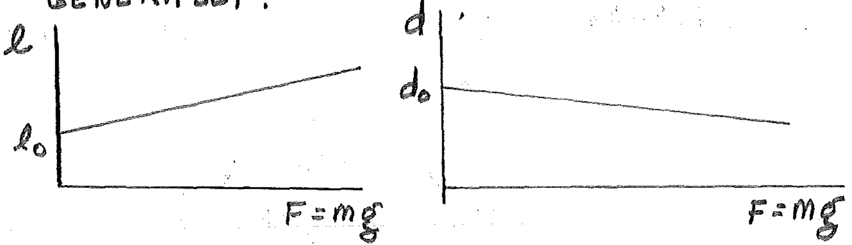
I) ELEMENTS OF ELASTICITY

A) STRESSES AND STRAINS

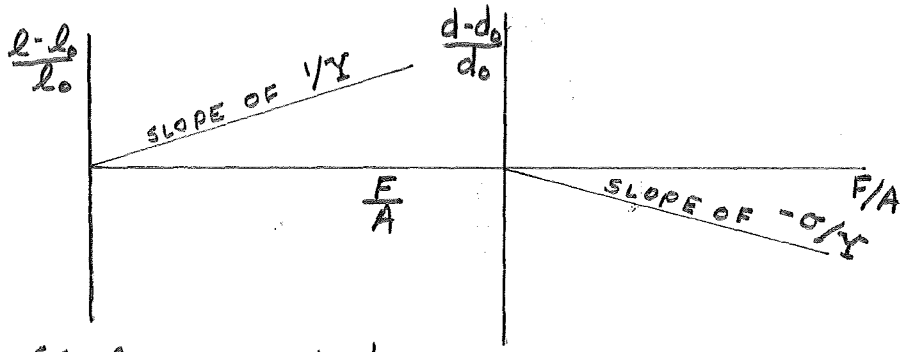
1) ON A STRING



GENERALLY:



MORE CONVENIENTLY



$\frac{l-l_0}{l_0}$ AND $\frac{d-d_0}{d_0}$ ARE STRAINS (DIMENSIONLESS)

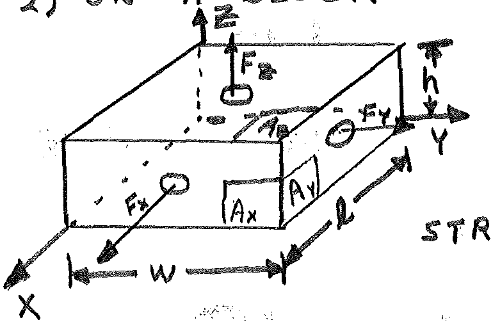
F/A IS THE STRESS (FORCE/AREA)

$$\frac{l-l_0}{l_0} = \frac{1}{Y}\frac{F}{A}$$

$$\frac{d-d_0}{d_0} = \frac{-\sigma}{Y}\frac{F}{A}$$

WHERE Y (YOUNG'S MODULUS) AND σ (POISSON'S RATIO) ARE SUFFICIENT TO COMPLETELY DESCRIBE THE ELASTIC BEHAVIOR OF HOMOGENEOUS ISOTROPIC MATERIALS.

2) ON A BLOCK



$$\begin{bmatrix}\epsilon_{XX}\\ \epsilon_{YY}\\ \epsilon_{ZZ}\end{bmatrix} = \begin{bmatrix}1/Y & -\sigma/Y & -\sigma/Y\\ -\sigma/Y & 1/Y & -\sigma/Y\\ -\sigma/Y & -\sigma/Y & 1/Y\end{bmatrix}\begin{bmatrix}S_{XX}\\ S_{YY}\\ S_{ZZ}\end{bmatrix}$$

STRAINS: ~~ϵ_{XX} = w~~

$\epsilon_{XX} = \frac{l-l_0}{l_0}$

$\epsilon_{YY} = \frac{w-w_0}{w_0}$

$\epsilon_{ZZ} = \frac{h-h_0}{w_0}$

STRESSES

$S_{XX} = F_X/A_X$

$S_{YY} = F_Y/A_Y$

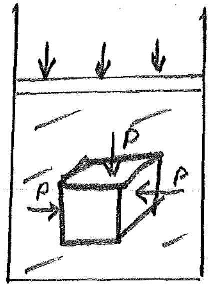
$S_{ZZ} = F_Z/A_Z$

B) BULK MODULUS

1) $B \overset{\Delta}{=} - \frac{\Delta P}{\Delta V / V}$

ISOTHERMAL BULK MODULUS COMPUTED @ CONSTANT TEMP.
ADIABATIC " " " " ZERO HEAT TRANSFE

2) RELATIONSHIP TO Y AND σ



V = VOLUME OF BLOCK @ PRESSURE p
V' = " " " " " p'

PRESSURE p
$S_{xx} = S_{yy} = S_{zz} = -p$

PRESSURE p'
$S_{xx} = S_{yy} = S_{zz} = -p'$

CHANGE IN STRAIN:

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \frac{1}{Y}(2\sigma - 1)(p - p')$$

CHANGE IN VOLUME: $\epsilon_{ii} = \frac{d - d_0}{d_0} \Rightarrow d = d_0(\epsilon_{ii} + 1)$

$$V' - V = l'w'h' - lwh$$

$$= l(\epsilon_{xx}+1)\,w(\epsilon_{xx}+1)\,h(\epsilon_{xx}+1) - lwh$$

$$= V\left[(1+\epsilon_{xx})^3 - 1\right]$$

$\epsilon_{xx} << 1$

$$\Rightarrow V' - V \simeq V\left[(1 + 3\epsilon_{xx}) - 1\right]$$

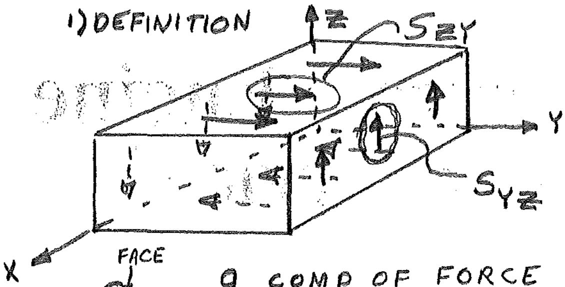
OR $\frac{V' - V}{V} = 3\epsilon_{xx} = \frac{3}{Y}(2\sigma - 1)(p - p')$

THUS: $B = \frac{-(p' - p)}{(V' - V)/V} = \frac{Y}{3(1 - 2\sigma)}$

IN THAN $B > 0$, $\sigma < \frac{1}{2}$

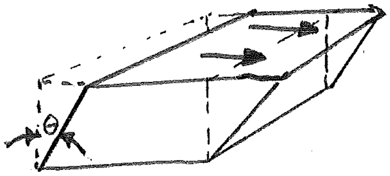
C) SHEARING STRESSES AND STRAINS, SHEAR MODULUS

1) DEFINITION



$$\overset{\text{FACE}}{S_{pq}} = \frac{q \text{ COMP OF FORCE ON } p \text{ FACE}}{\text{AREA OF } p \text{ FACE}} = \text{SHEARING STRESS}$$

WILL DEFORM THE BLOCK:

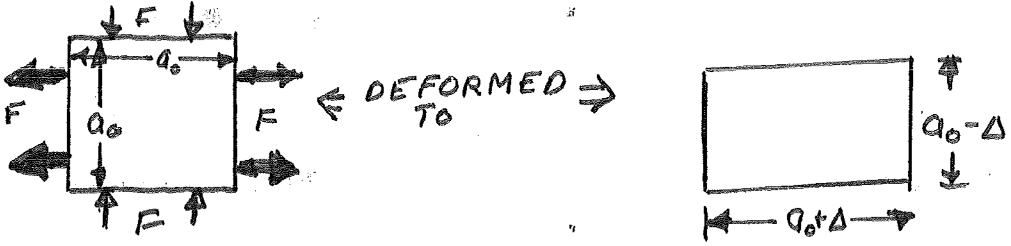


FOR S_{YZ}

$$G = \frac{S_{YZ}}{\theta} = \text{SHEAR MODULUS} \gg 1$$

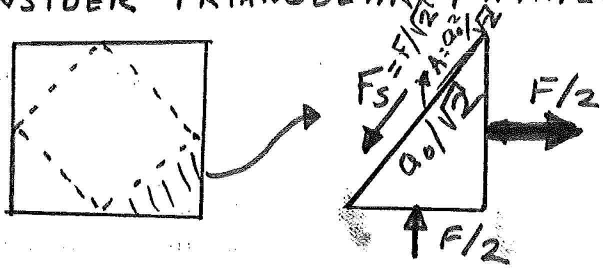
2) SHEAR MODULUS RELATIONSHIP TO Y AND σ

CONSIDER CUBE OF VOLUME a_0^2, AND Y & Z FORCES

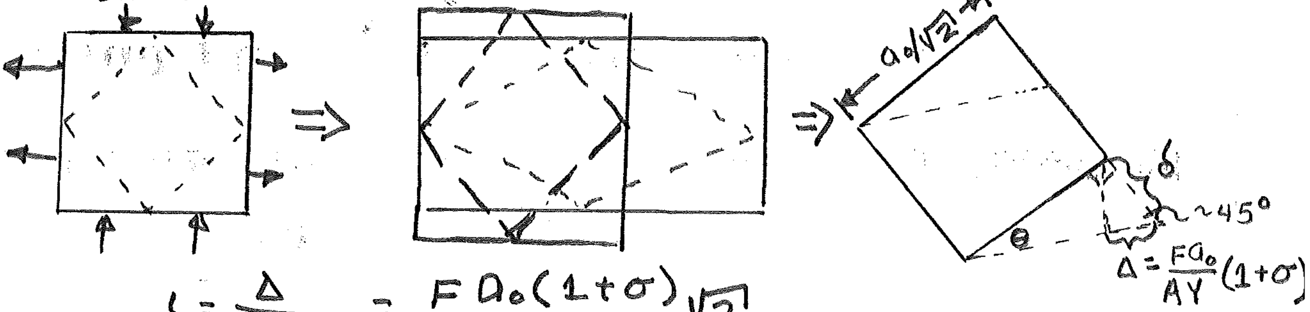


$$\Delta = \frac{F a_0}{A Y}(1 + \sigma)$$

CONSIDER TRIANGULAR PARALLELPIPED:



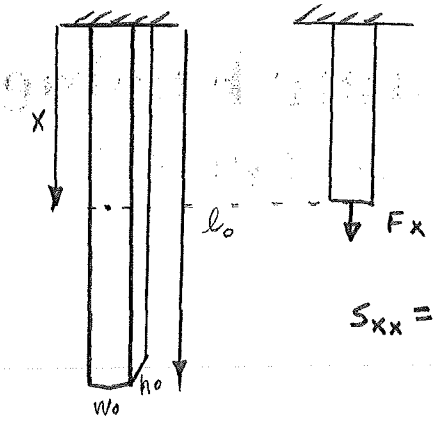
SHEARING STRESS = F



$$\delta = \frac{\Delta}{\cos 45°} = \frac{F a_0 (1 + \sigma)}{A Y}\sqrt{2}$$

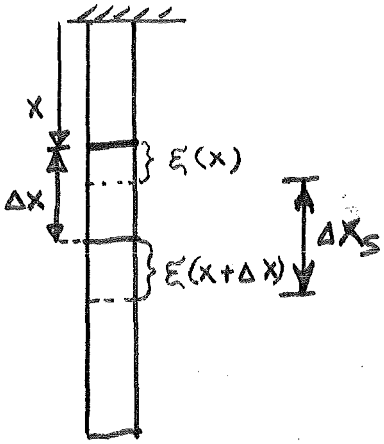
$$\Rightarrow \theta \doteq \frac{\delta}{a_0/\sqrt{2}} = \frac{F(1+\sigma)\,2}{A Y} \Rightarrow \frac{F/A}{\theta} = \frac{F_s/A_s}{\theta} = G = \frac{Y}{2(1+\sigma)}$$

D) STRESS AND STRAIN AT A POINT



$$F_x = mg - \rho\,[h_0 w_0 x]\,g$$

$$S_{xx} = \frac{F_x}{h_0 w_0} = \frac{mg}{h_0 w_0} - \rho g x$$
$$= \rho g (l_0 - x)$$



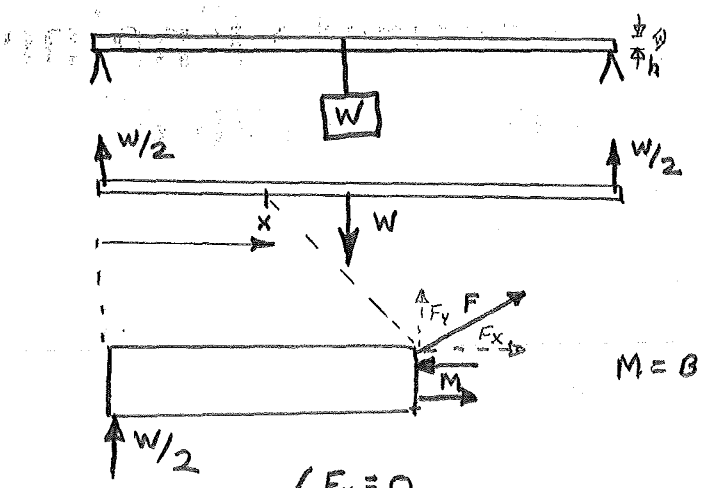
$$\epsilon_{xx} \triangleq \lim_{\Delta x \to 0} \frac{\Delta x_s - \Delta x}{\Delta x}$$

$$\Delta x_s - \Delta x = \xi(x+\Delta x) - \xi(x)$$
$$\Rightarrow \epsilon_{xx} = \lim_{\Delta x \to 0} \frac{\xi(x+\Delta x) - \xi(x)}{\Delta x} = \frac{d\xi}{dx}$$

NOW $\epsilon_{xx} = \frac{1}{Y} S_{xx} = \frac{1}{Y} \rho g (l_0 - x) = \frac{d\xi}{dx}$

$$\Rightarrow \xi(x) = \frac{1}{Y} \rho g \left[l_0 x - \frac{x^2}{2}\right]$$

E) THIN BEAM



M = BENDING MOMENT (TORQUE IN Z DIRECTION)

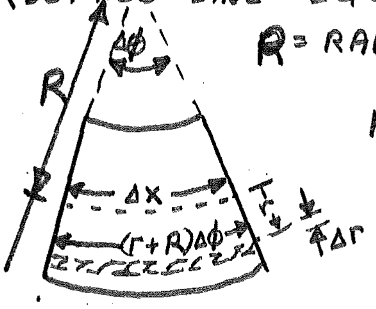
$$\begin{cases} F_x = 0 \\ F_y = -W/2 \\ M = \frac{Wx}{2} \end{cases}$$

THUS:

$$S_{xx} = \frac{F_x}{wh} = 0$$

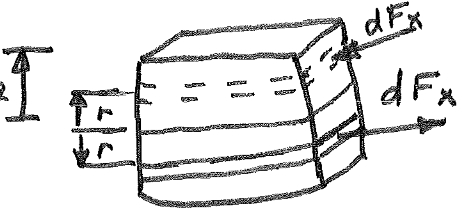
$$S_{yy} = \frac{F_y}{wh} = \frac{-W}{2wh}$$

CONSIDER Δx OF THE ROD, @ DISTANCE X:
(DOTTED LINE EQUAL IN LENGTH TO UNDISTORTED ROD)



R = RADIUS OF CURVATURE = $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$

$\simeq \left[\frac{d^2y}{dx^2}\right]^{-1}$ FOR $\frac{dy}{dx} \ll 1$



CHANGE IN LENGTH OF SHADED STRIP IS:

$$(r+R)\Delta\phi - R\Delta\phi = r\Delta\phi$$

STRAIN ON STRIP:

$$\frac{(r+R)\Delta\phi - R\Delta\phi}{R\Delta\phi} = \frac{r}{R}$$

STRESS AT STRIP:

$$S_{xx} = Y\epsilon_{xx} = Yr/R = \frac{dF_x}{w\,dr}$$

$$\Rightarrow dF_x = \frac{Yrw}{R}\,dr$$

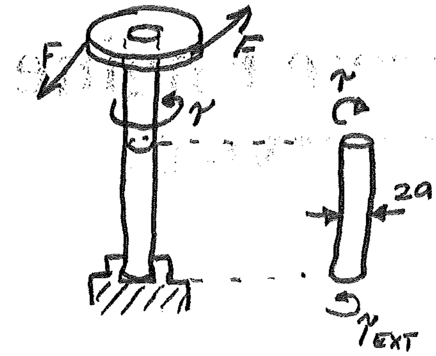
TORQUE DUE TO dF_x AND dF_y: $d\gamma_z = 2r\,dF_x = 2r^2\frac{Y\omega}{R}dr$

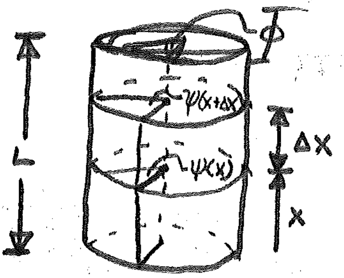
$$\Rightarrow M = \int_0^{h/2} 2\frac{Yr^2}{R}\omega\,dr = \frac{Y\omega h^3}{12R} = \frac{Ywh^3}{12}\frac{dY}{dx^2}$$

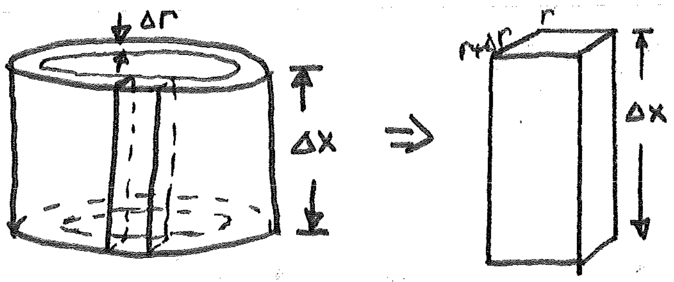
OR $\frac{d^2y}{dx^2} = \frac{12}{Ywh^3}M = \frac{12}{Ywh^3}\frac{W}{2}x$

$$\Rightarrow Y = \frac{W}{Ywh^3}\left(x^3 - \frac{3}{4}L^2x\right)$$

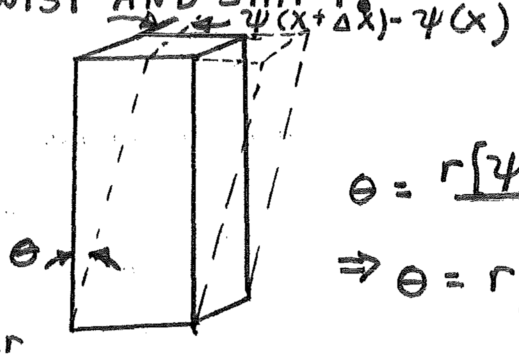
F) ROD UNDER TORSION





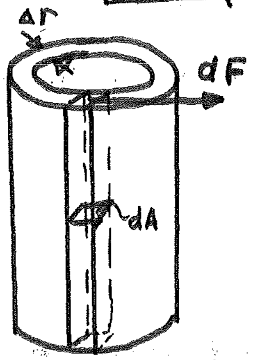


TWIST AND SHIFT:



$$\theta = \frac{r[\psi(x+\Delta x) - \psi(x)]}{\Delta x}$$

$$\Rightarrow \theta = r\frac{d\psi}{dx}$$



$$d\tau = r\,dF = rG\theta\,dA = r\left[Gr\frac{d\psi}{dr}\right]dA$$

$$\Rightarrow \cancel{\Delta\tau = d\tau\,\Delta r} \Rightarrow \Delta\tau = r\left[Gr\frac{d\psi}{dx}\right]2\pi r\Delta r$$

$$\cancel{= r\left[Gr\frac{d\psi}{dr}\right]dA}$$

$$\tau = \int_0^a r\left[Gr\frac{d\psi}{\delta x}\right]2\pi r\,dr = \frac{Ga^4\pi}{2}\frac{d\psi}{dx} = \tau_{EXT}$$

$$\Rightarrow \psi = \frac{\Phi}{L}x$$

$$\tau_{EXT} = \frac{Ga^4\pi}{2}\frac{\Phi}{L}$$

G) GENERALIZED CONCEPT OF STRAIN

$$d\xi = \frac{\delta \xi}{\delta X}dX + \frac{\delta \xi}{dY}dY + \frac{\delta \xi}{\delta Z}dZ$$

$$d\eta = \frac{\delta \eta}{\delta X}dX + \frac{d\eta}{dY}dY + \frac{\delta \eta}{\delta Z}dZ$$

$$d\zeta = \frac{\delta \zeta}{\delta X}dX + \frac{\delta \zeta}{\delta Y}dY + \frac{\delta \zeta}{\delta Z}dZ$$

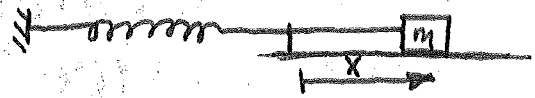
$$\epsilon_{XX} = \frac{\delta \xi}{\delta X} \qquad \epsilon_{XY} = \epsilon_{YX} = \frac{1}{2}\left(\frac{\delta \xi}{\delta Y} + \frac{\delta \eta}{\delta X}\right) =$$

$$\epsilon_{YY} = \frac{\delta \eta}{\delta Y} \qquad \epsilon_{XZ} = \epsilon_{ZX} = \frac{1}{2}\left(\frac{\delta \xi}{\delta Z} + \frac{\delta \zeta}{\delta X}\right)$$

$$\epsilon_{ZZ} = \frac{\delta \zeta}{\delta Z} \qquad \epsilon_{YZ} = \epsilon_{ZY} = \frac{1}{2}\left(\frac{\delta \eta}{\delta Z} + \frac{\delta \zeta}{\delta Y}\right)$$

II) HARMONIC MOTION

A) THE SIMPLE HARMONIC OSCILLATOR



$$-Kx = m\ddot{x}$$

$$\omega_0 = \sqrt{K/M}$$

$$\Rightarrow \ddot{x} + \omega_0^2 x = 0$$

ASSUME: $x(t) = \sum_{n=0}^{\infty} a_n t^n$

(PLUG AND CHUG)

$$x(t) = C\cos\omega_0 t + D\sin\omega_0 t$$

B) COMPLEX FORM OF SOLUTION

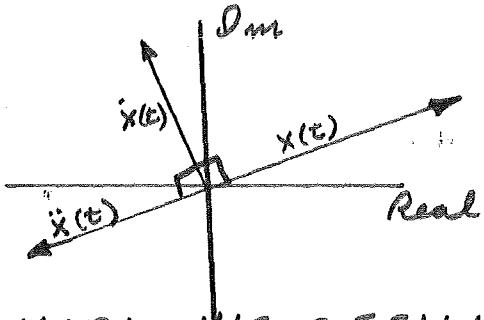
$$x(t) = Ae^{i\omega_0 t}$$

C) VELOCITY, ACCELERATION, AND PHASE

$$x(t) = Ae^{i\omega_0 t}$$

$$\dot{x}(t) = i\omega A e^{i\omega_0 t}$$

$$\ddot{x}(t) = -\omega^2 A e^{i\omega_0 t}$$



D) ENERGY OF SIMPLE HARMONIC OSCILLATOR

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}mA^2\omega_0^2$$

E) DAMPED HARMONIC MOTION

$$m\ddot{x} + R\dot{x} + kx = 0$$

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0 \ni \omega_0 = \sqrt{\frac{K}{m}} \; ; \; \alpha = R/2m$$

$$\Rightarrow x = e^{-\alpha t}[A\cos(\omega_0 t + \phi)] \ni \omega_D = \sqrt{\omega_0^2 - \alpha^2}$$

F) DRIVEN HARMONIC OSCILLATOR

$$m\ddot{x} + R\dot{x} + KX = F_0\cos\omega t$$

PARTICULAR SOLUTION

$$X = \frac{F_0/\omega\,(\sin\omega t - \Theta)}{|Z_m|} \; ; \; Z_m = R + i(\omega m - K/\omega)$$

$$\Theta = \tan^{-1}\left(\frac{\omega m - K/\omega}{R}\right)$$

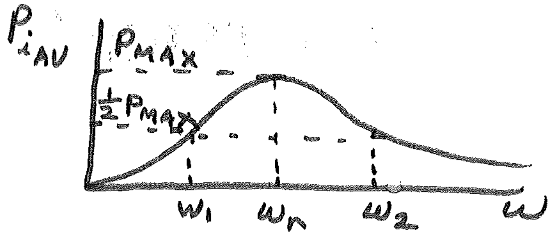
GENERAL SOLUTION:

$$X = Ae^{-\alpha t}\cos(\omega_D t + \phi) + \frac{(F_0/m)(\sin(\omega t - \phi))}{\sqrt{R^2 + (\omega m - \frac{K}{\omega})^2}}$$

x(t) LAGS $F\cos(\omega t)$ BY Θ

G) MECHANICAL RESONANCE

$$P_{iAV} = \frac{F_0^2 R}{2[R^2+(\omega m - \frac{k}{\omega})^2]}$$



$$\omega_r = \sqrt{K/M}$$

$$Q = \frac{\omega_r}{\omega_2-\omega_1} = \frac{1}{R}\sqrt{KM}$$

$$\omega_2 = \frac{R}{2m} + \sqrt{(R/2m)^2 + K/M}$$

$$\omega_1 = \frac{R}{2m} - \sqrt{(R/2m^2) + K/M}$$

H) COMPLEX FORM OF SOLUTION OF DRIVEN OSCILLATOR

$$m\ddot{x} + R\dot{x} + kx = F_0 \sin \omega t$$

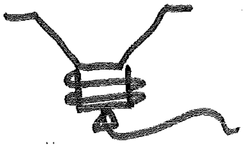
$$\Rightarrow x(t) = \frac{-iF_0/\omega}{Z_m} e^{i\omega t}$$

$x(t)$ LAGS $F_0 \sin \omega t$ BY $\tan^{-1}\left(\frac{\omega m - \frac{k}{\omega}}{R}\right)$

I) MECHANICAL IMPEDANCE

$$Z_m = R + i(\omega m - \frac{k}{\omega})$$

J) THE LOUDSPEAKER AS A DRIVEN DAMPED OSCILLATOR



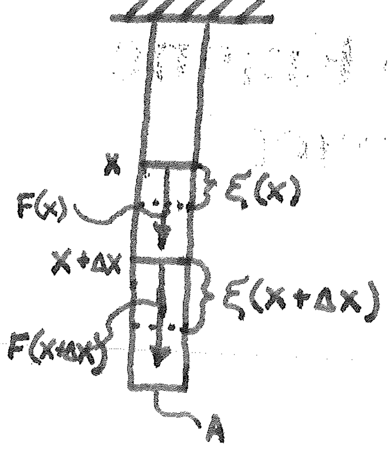
$$m\ddot{y} + R\dot{y} + ky = B l I_0 \cos \omega t$$

$$\Rightarrow \dot{y} = \frac{B l I_0 e^{i\omega t}}{Z_m}$$

ROD WAVES

LONGITUDINAL WAVES ON RODS

A) DERIVATION



$$\epsilon_{xx} = \frac{\delta\xi}{\delta x} = \frac{1}{Y} S_{xx} = \frac{F_x}{AY}$$

$$\Rightarrow F_x = YA\frac{\delta\xi}{\delta x}$$

$$F(x+\Delta x) - F(x) = \rho A \Delta x \left.\frac{\delta^2\xi}{\delta t^2}\right|_{x+\Delta x}$$

$$\Rightarrow \frac{dF_x}{\delta x} = \rho A \frac{\delta^2\xi}{\delta t^2}$$

$$\frac{\delta F_x}{\delta x} = YA\frac{\delta^2\xi}{\delta x^2} = \rho A \frac{\delta^2\xi}{\delta t^2}$$

$$\therefore c^2\frac{\delta^2\xi}{\delta x^2} = \frac{\delta^2\xi}{\delta t^2} \ni c = \sqrt{Y/\rho}$$

B) SOLUTION

$$\xi(x,t) = X(x)\,H(t)$$

$$\Rightarrow c^2\frac{d^2X(x)}{dx^2}H(t) = X(x)\frac{d^2H(t)}{dt^2}$$

$$\frac{c^2}{X}\frac{d^2X(x)}{dx^2} = \frac{1}{H}\frac{d^2H(t)}{dt^2} = -\omega^2$$

$$\frac{d^2H}{dt^2} = -Hc^2 \Rightarrow H = a_1\cos\omega t + b_1\sin\omega t$$

$$\frac{d^2X}{dx^2} = \cancel{(\omega)}-\left(\frac{\omega}{c}\right)^2 X = X = a_2\cos\left(\frac{\omega}{c}\right)x + b_2\sin\left(\frac{\omega}{c}\right)x$$

$$\xi(x,t) = \left[c_1\cos\left(\frac{\omega}{c}\right)x + c_2\sin\left(\frac{\omega}{c}\right)x\right]\cos\omega t$$
$$+\left[c_3\cos\left(\frac{\omega}{c}\right)x + c_4\sin\left(\frac{\omega}{c}\right)x\right]\sin\omega t$$

C) BOUNDRY CONDITIONS

1) FREE END

~~$\xi(L,t)$ or $\xi(0,t)=$~~

$$F_x(\overset{0}{L},t) = 0 \Rightarrow \left.\frac{\delta\xi}{\delta x}\right|_{\overset{0}{L},t} = 0$$

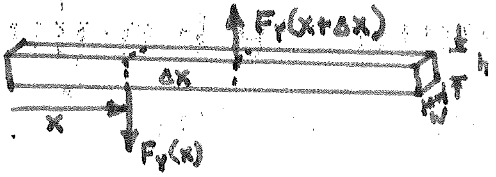
2) FIXED END

$$\xi(\overset{0}{L},t) = 0$$

ROD WAVES

II) TRANSVERSE WAVES IN RODS

Ⓐ DERIVATION



$$F_Y(x+\Delta x) - F_Y(x) = \rho w h \Delta x \left.\frac{\delta^2 y}{\delta t^2}\right|_{x+\Delta x/2}$$

$$\Rightarrow \frac{\delta F_x}{\delta x} = \rho w h \cancel{\Delta x} \frac{\delta^2 y}{\delta t^2}$$

PREVIOUSLY DERIVED:

$$\mathcal{M} = \frac{Y w h^3}{12}\frac{\delta^2 y}{\delta x^2}\ ; \ F_Y = -\frac{\delta \mathcal{M}}{\delta x}$$

$$\Rightarrow \frac{dF_Y}{\delta x} = -\frac{\delta^2 \mathcal{M}}{\delta x^2} = -\frac{Y w h^3}{12}\frac{d^4 y}{d x^4}$$

COMBINING

$$\frac{dF_x}{dx} = -\frac{Y w h^3}{12}\frac{d^4 y}{dx^4} = \rho w h \frac{\delta^2 y}{\delta t^2}$$

$$-(cI)^2 \frac{\delta^4 y}{\delta x^4} = \frac{\delta^2 y}{\delta t^2} \quad \ni c = \sqrt{Y/\rho}\ ; \ I = \frac{h}{\sqrt{12}}$$

Ⓑ SOLUTION

$$Y(x,t) = X(x)\,H(t)$$

$$-(cI)^2 \frac{d^4 X}{dx^4} H = X \frac{d^2 H}{dt^2}$$

$$-(cI)^2 \frac{1}{X}\frac{d^4 X}{dx^4} = \frac{1}{H}\frac{d^2 H}{dt^2} = -\omega^2$$

$$\frac{d^2 H}{dt^2} = -\omega^2 H \Rightarrow H(t) = a_1 \cos \omega t + a_2 \sin \omega t$$

$$\frac{d^4 X}{dx^2} = -\frac{\omega^2}{(cI)^2} X = -\alpha^2 X$$

$$\Rightarrow X(x) = b_1 \cos \alpha x + b_2 \sin \alpha x + b_3 \cosh \alpha x + b_4 \sinh \alpha x$$

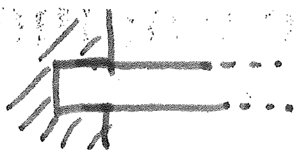
$$\Rightarrow Y(x,t) = [A_1 \cos \alpha x + A_2 \sin \alpha x + B_3 \cosh \alpha x + A_4 \sinh \alpha x] \cos \omega t$$

$$+ [B_1 \cos \alpha x + B_2 \sin \alpha x + B_3 \cosh \alpha x + B_4 \sinh \alpha x] \sin \omega t$$

ROD WAVES

ⓒ BOUNDRY CONDITIONS

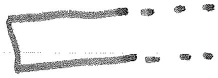
① CLAMPED END



$$Y(^{0}_{L},t)=0$$

$$\left.\frac{\delta Y}{\delta X}\right|_{^{0}_{L},t}=0$$

② FREE END



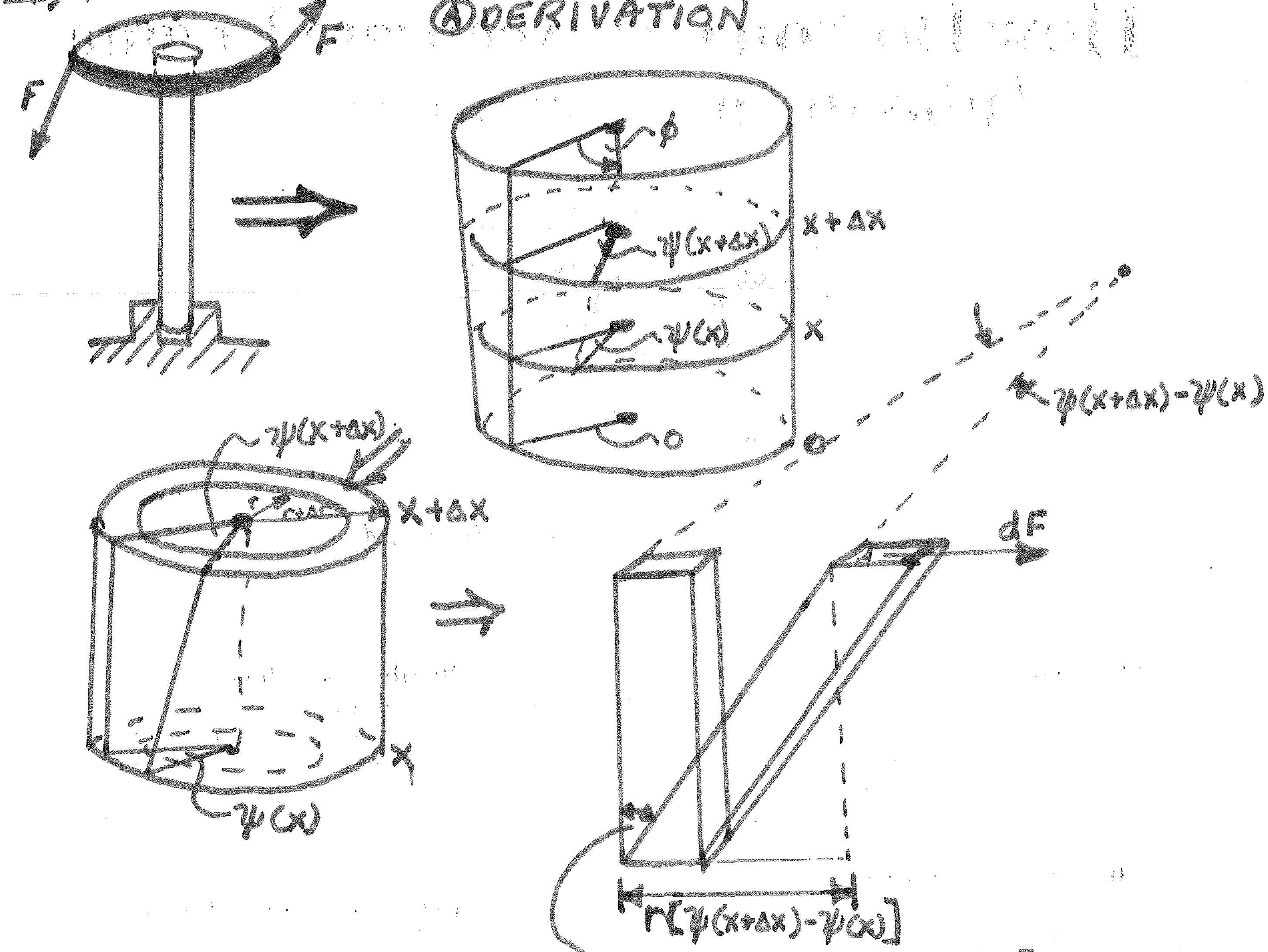
$$\mathcal{M}=0 \Rightarrow \left.\frac{\delta^2 y}{\delta x^2}\right|_{^{0}_{L},t}=0$$

$$F_Y=0 \Rightarrow \left.\frac{\delta^3 Y}{\delta X^3}\right|_{^{0}_{L},t}=0$$

ROD WAVES

III) TORSIONAL WAVES IN RODS

(A) DERIVATION



$$\theta = \frac{r[\psi(x+\Delta x)-\psi(x)]}{\Delta x} = r\frac{d\psi}{dx}$$

$$\frac{dF}{dA} = G\theta \Rightarrow dF = G\theta\, dA = G r \frac{\delta\psi}{\delta x} dA$$

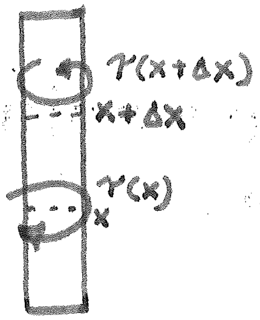
THE TORQUE ABOUT A SHELL:

$$\Delta\gamma = G r \frac{\delta\psi}{\delta x}(2\pi r\, dr)$$

$$\Rightarrow \gamma = \int_0^a G r\left(\frac{\delta\psi}{\delta x} 2\pi r\right) dr$$

$$= G \frac{\delta\psi}{\delta x} \frac{\pi a^4}{2}$$

ROD WAVES



$$\tau(x+\Delta x)-\tau(x)=\tfrac{1}{2}I\alpha_x$$
$$=\left[\tfrac{1}{2}\rho\,\Delta x\,\pi a^2\right]\frac{\delta^2\psi}{\delta t^2}$$
$$\Rightarrow \frac{\delta\tau}{\delta x}=\frac{\pi a^4\rho}{2}\,\frac{\delta^2\psi}{\delta t^2}$$

$$\frac{\delta\tau}{\delta x}=\frac{\pi a^4\rho}{2}\,\frac{\delta^2\psi}{\delta t^2}=G\frac{\delta^2\psi}{\delta x^2}\,\frac{\pi a^4}{2}$$
$$\Rightarrow c^2\,\frac{\delta^2\psi}{\delta x^2}=\frac{\delta^2\psi}{\delta t^2}\ \ni\ c=\sqrt{G/\rho}$$

Ⓑ SOLUTION

SAME AS FOR LONGITUDINAL WAVES, REPLACING ψ FOR ξ

Ⓒ BOUNDRY CONDITIONS

1) FREE END:

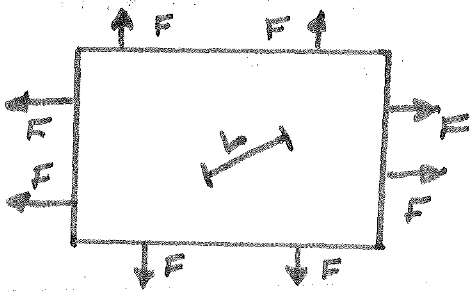
$$\tau=0\Rightarrow \left.\frac{\delta\psi}{\delta x}\right|_{\substack{0\\L},t}=0$$

2) CLAMPED END:

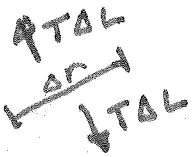
$$\psi\left(\substack{0\\L},t\right)=0$$

MEMBRANE WAVES

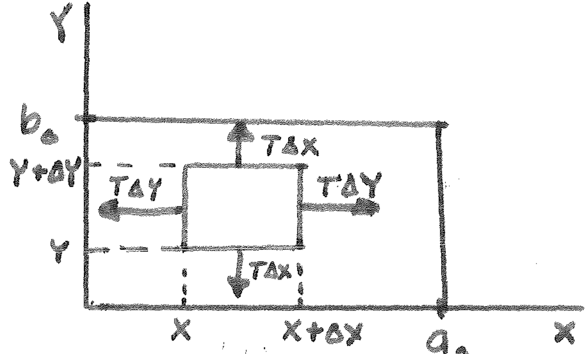
I) RECTANGULAR PERIMETER

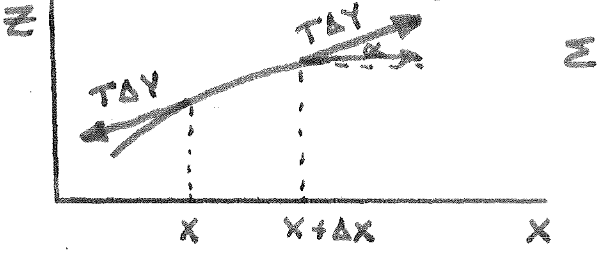


$$T = \frac{\Sigma |F|}{\text{PERIMETER}}$$

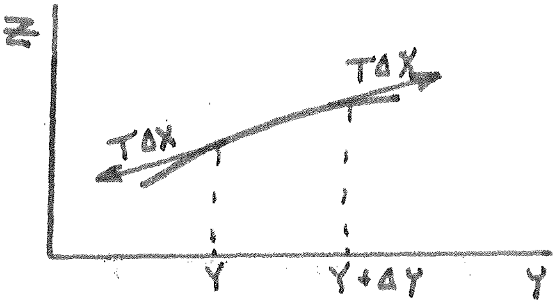


Ⓐ DERIVATION





$$\Sigma F_x = T\Delta Y \left.\frac{\delta Z}{\delta X}\right|_{X+\Delta X} - T\Delta Y \left.\frac{\delta Z}{\delta X}\right|_{X}$$



$$\Sigma F_Y = T\Delta X \left.\frac{\delta Z}{\delta Y}\right|_{Y+\Delta Y} - T\Delta X \left.\frac{\delta Z}{\delta Y}\right|_{Y}$$

$$T\left[\Delta X\left\{\left.\frac{\delta Z}{\delta Y}\right|_{Y+\Delta Y} - \left.\frac{\delta Z}{\delta Y}\right|_{Y}\right\} + \Delta Y\left\{\left.\frac{\delta Z}{\delta X}\right|_{X+\Delta X} - \left.\frac{\delta Z}{\delta X}\right|_{X}\right\}\right] = \sigma \Delta X \Delta Y \frac{\delta^2 Z}{\delta t^2}$$

$$\Rightarrow T\left[\frac{\delta^2 Z}{\delta Y^2} + \frac{\delta^2 Z}{\delta X^2}\right] = \sigma \frac{\delta^2 Z}{\delta t^2}$$

$$\therefore\ c^2\left[\frac{\delta^2 Z}{\delta X^2} + \frac{\delta^2 Z}{\delta Y^2}\right] = \frac{\delta^2 Z}{\delta t^2} \Rightarrow c = \sqrt{T/\sigma}$$

ⓑ SOLUTION

$$Z(x,y,t) = \mathbb{X}(x)\,\mathbb{Y}(y)\,H(t)$$

$$\Rightarrow c^2\left[\mathbb{Y}H\frac{d^2\mathbb{X}}{dx^2} + \mathbb{X}H\frac{d^2\mathbb{Y}}{dy^2}\right] = \mathbb{X}\mathbb{Y}\frac{d^2H}{dt^2}$$

$$c^2\left[\frac{1}{\mathbb{X}}\frac{d^2\mathbb{X}}{dx^2} + \frac{1}{\mathbb{Y}}\frac{d^2\mathbb{Y}}{dy^2}\right] = \frac{1}{H}\frac{d^2H}{dt^2} = -\omega^2$$

$$\frac{d^2H}{dt^2} = -\omega^2 H \Rightarrow d_1\cos\omega t + d^2\sin\omega t$$

$$\frac{1}{\mathbb{X}}\frac{d^2\mathbb{X}}{dx^2} + \frac{1}{\mathbb{Y}}\frac{d^2\mathbb{Y}}{dy^2} = -\left(\frac{\omega}{c}\right)^2$$

$$\frac{1}{\mathbb{X}}\frac{d^2\mathbb{X}}{dx^2} = -\left(\frac{\omega}{c}\right)^2 - \frac{1}{\mathbb{Y}}\frac{d^2\mathbb{Y}}{dy^2} = -\alpha^2$$

$$\frac{d^2\mathbb{X}}{dx^2} = -\alpha^2\mathbb{X} \Rightarrow \mathbb{X} = d_3\cos\alpha x + d_4\sin\alpha x$$

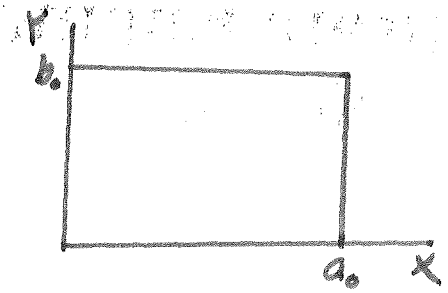
~~$\frac{d^2\mathbb{Y}}{dy^2} = \left[\left(\frac{\omega}{c}\right)^2\right.$~~

$$\frac{d^2\mathbb{Y}}{dy^2} = -\left[\left(\frac{\omega}{c}\right)^2 - \alpha^2\right]\mathbb{Y}$$

$$\Rightarrow \mathbb{Y} = d_5\cos\left[\sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\,y\right] + d_6\sin\left[\sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\,y\right]$$

$\therefore Z(x,y,t) = H(t)\,\mathbb{X}(x)\,\mathbb{Y}(y)$ FROM ABOVE

Ⓒ APPLICATION OF BOUNDRY CONDITIONS



$Z(0,Y,t)=0 \Rightarrow d_3=0$

$Z(X,0,t)=0 \Rightarrow d_5=0$

$Z(a_0,Y,t)=0$

$Z(X,b_0,t)=0$

$$Z(X,Y,t) = \sin\alpha X \sin\sqrt{\left(\frac{\omega}{c}\right)^2-\alpha^2}\,Y\left[A\cos\omega t + B\sin\omega t\right]$$

$$Z(a_0,Y,t)=0 \Rightarrow \sin\alpha a_0 = 0 \Rightarrow \alpha a_0 = m\pi \qquad ; m = 0,2,3,\ldots$$

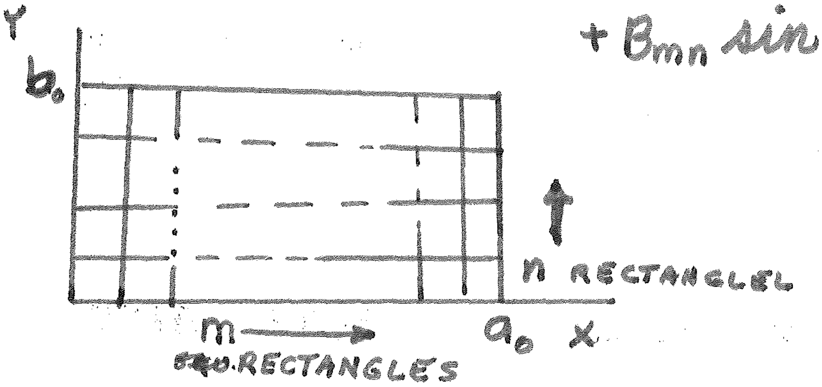
$$\Rightarrow \alpha = \frac{m\pi}{a_0}$$

$$Z(X,b_0,t)=0 \Rightarrow \sin\sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{m\pi}{a_0}\right)^2}\,b = 0$$

$$\Rightarrow \sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{m\pi}{a_0}\right)^2} = \frac{n\pi}{b_0} \qquad ; n = 1,2,3,\ldots$$

$$\omega_{mn} = c\sqrt{\left(\frac{n\pi}{b_0}\right)^2+\left(\frac{m\pi}{a_0}\right)^2} = c\pi\sqrt{\left(\frac{n}{b_0}\right)^2+\left(\frac{m}{a_0}\right)^2}$$

$$\therefore Z_{mn}(X,Y,t) = \sin\frac{m\pi}{a_0}X \sin\frac{n\pi}{b_0}Y\left[A_{mn}\cos\pi c\sqrt{\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2}\,t + B_{mn}\sin\pi c\sqrt{\left(\frac{n}{b}\right)^2+\left(\frac{m}{a}\right)^2}\,t\right]$$



Ⓓ FOURIER EXPANSION

$$Z(x,Y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Z_{mn}(x,Y,t)$$

GIVEN: $Z(x,Y,0)$ AND $V(x,Y,0) = \left.\frac{dZ}{dt}\right|_{x,Y,0}$

$$\Rightarrow Z_0(x,Y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}Y\, A_{mn}$$

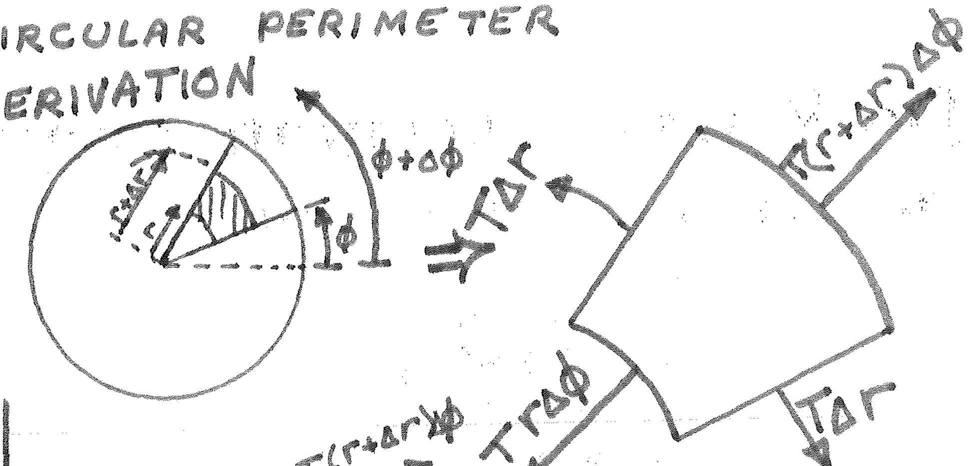
$$\cancel{\int_{-\infty}^{\infty} Z_0 \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}Y =}$$

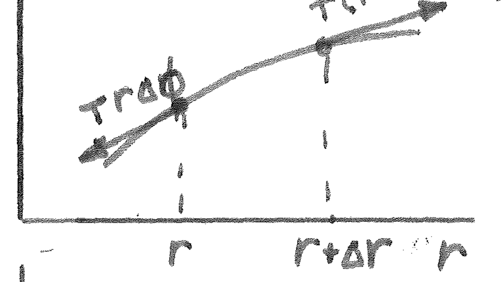
$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a Z_0(x,Y)\sin\frac{m\pi x}{a}\sin\frac{n\pi Y}{b}\,dx\,dY$$

$$B_{mn} \overset{?}{=} \frac{-4c\pi}{ab}\int_0^b\int_0^a V_0(x,Y)\sin\frac{m\pi x}{a}\sin\frac{n\pi Y}{b}\,dx\,dY$$

II) CIRCULAR PERIMETER

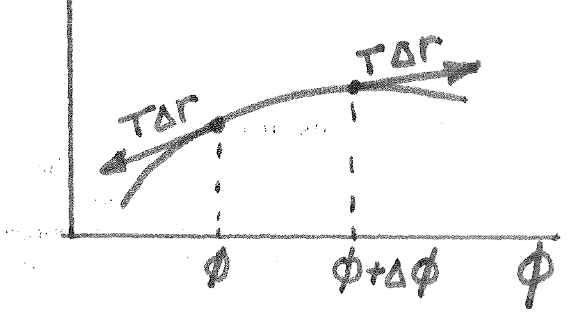
A) DERIVATION





~~$\Sigma F_r = T(r+\Delta r)\Delta\phi - Tr\Delta\phi$~~

$$\Sigma F_r = T(r+\Delta r)\Delta\phi \frac{\delta z}{\delta r}\Big|_{r+\Delta r} - Tr\Delta\phi\frac{\delta z}{\delta r}\Big|_{r}$$



$$\Sigma F_\phi = T\Delta r \frac{1}{r}\frac{\delta z}{\delta\phi}\Big|_{\phi+\Delta\phi} - T\Delta r\frac{\delta z}{r\,\delta\phi}\Big|_{\phi}$$

~~$\Sigma F = T\Big[\Delta\phi\Big\{(r+\Delta r)\frac{\delta z}{\delta r}\Big|_{r+\Delta r} - r\frac{\delta z}{\delta r}\Big|_{r}\Big] + \frac{\Delta r}{r}\Big\{$~~

$$\Sigma F = T\Big[\Delta\phi\Big\{(r+\Delta r)\frac{\delta z}{\delta r}\Big|_{r+\Delta r} - r\frac{\delta z}{\delta r}\Big|_{r}\Big\} + \frac{\Delta r}{r}\Big\{\frac{\delta z}{\delta\phi}\Big|_{\phi+\Delta\phi} - \frac{\delta z}{\delta\phi}\Big|_{\phi}\Big\}\Big] = \sigma\Delta r\, r\Delta\phi\frac{\delta^2 z}{\delta t^2}\Big|_{r\phi t}$$

$$\Rightarrow T\Big[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r}\Big] + \frac{T}{r^2}\frac{\delta^2 z}{\delta\phi^2} = \sigma\frac{\delta^2 z}{\delta t^2}$$

$$\therefore c^2\Big[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r} + \frac{1}{r^2}\frac{\delta^2 z}{\delta\phi^2}\Big] = \frac{\delta^2 z}{\delta t^2} \ni c = \sqrt{T/\sigma}$$

⑬ SOLUTION

$$Z(r,\phi,t) = R(r)\Phi(\phi)H(t)$$

$$c^2\left[\Phi H\frac{d^2R}{dr^2} + \frac{1}{r}\phi H\frac{dR}{dr} + \frac{1}{r^2}RH\frac{d^2\phi}{d\phi}\right] = R\phi\frac{d^2H}{dt^2}$$

$$c^2\left[\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\phi}\frac{d^2\phi}{d\phi}\right] = \frac{1}{H}\frac{d^2H}{dt^2} = -\omega^2$$

$$\frac{d^2H}{dt^2} = -\omega^2 H \rightarrow H(t) = d_1\cos\omega t + d_2\sin\omega t$$

$$\cancel{\left[\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr}\right]}$$

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \frac{1}{r^2\phi}\frac{d^2\phi}{d\phi^2} = -\left(\frac{\omega}{c}\right)^2$$

$$r^2\left[\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + \left(\frac{\omega}{c}\right)^2\right] = -\frac{1}{\Phi}\frac{d^2\phi}{d\phi^2} = m^2$$

$$\frac{d^2\phi}{d\phi^2} = -m^2\Phi \Rightarrow \Phi(\phi) = d_3\cos m\phi + d_4\sin m\phi$$

(NOTE, $\Phi(\phi) = \Phi(\phi + 2n\pi) \Rightarrow m \in$ INTEGER)

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \left(\frac{m}{r}\right)^2\right)R = 0 \quad \ni k = \sqrt{\frac{\omega}{c}}$$

LET $R = \sum_{n=0}^{\infty} a_n r^n$

$$-R\left(\frac{m}{r}\right)^2 = -\left(\frac{m}{r}\right)^2 a_0 - \frac{m^2}{r}a_1 + m^2a_2 - m^2ra_3 + m^2ra_4 + \dots$$

$$k^2R = k^2a_0 + k^2ra_1 + k^2r^2a_2 + \dots$$

$$\frac{1}{r}\frac{dR}{dr} = \frac{a_1}{r} + 2a_2 + 3ra_3 + 4r^2a_4 + \dots$$

$$\frac{d^2R}{dr} = 2a_2 + 6ra_3 + 12r^2a_4 + \dots$$

$a_1 = 0 \Rightarrow a_{2p+1} = 0$

$a_2 = -\frac{k^2}{4}a_0$

$a_4 = -\frac{k^2}{16}a_2 = \frac{k^4}{(4)(16)}a_0$

$\vdots$

$$\Rightarrow R(r) = a_0\left[1 - \frac{kr^2}{4} + \frac{kr^4}{(4)(16} - \dots\right]$$

ⓒ BOUNDRY CONDITIONS

$$z(r,\phi,t) = J_m\left(\frac{\omega}{c}r\right)\left[d_3 \cos m\phi + d_4 \sin m\phi\right]$$
$$\left[d_1 \cos \omega t + d_2 \sin \omega t\right]$$

$$z(a,\phi,t) = 0$$

$$\Rightarrow J_m\left(\frac{\omega}{c}a\right) = 0$$

FOR $m=0$

$$\frac{\omega}{c}a = 2.405,\ 5.52,\ 8.65$$

$$\Rightarrow \omega_{0n} = \frac{2.405c}{a},\ \frac{5.52c}{a},\ \frac{8.65c}{a}$$

FOR $m=1$

$$\frac{\omega}{c}a = 3.83,\ 7.01$$

$$\Rightarrow \omega_{1n} = \frac{3.83c}{a},\ \frac{7.01c}{a},\ \ldots$$

FOR $m=2$

$$\frac{\omega}{c}a = 5.15,\ 8.41,\ \ldots$$

$$\Rightarrow \omega_{2n} = \frac{5.15c}{a},\ \frac{8.41c}{a},\ \ldots$$

$$Z_{01} = C_{01} J_0\left(\frac{2.405r}{a}\right)\cos\left(\frac{2.405c}{a}t + \Omega_{01}\right)$$

$$Z_{02} = C_{02} J_0\left(\frac{5.52r}{a}\right)\cos\left(\frac{5.52}{a}ct + \Omega_{02}\right)$$

$$\vdots$$

$$Z_{11} = C_{11} J_1\left(\frac{3.83r}{a}\right)\cos(\phi + \phi_{11})\cos\left(\frac{3.83}{a}ct + \Omega_{11}\right)$$

$$Z_{12} = C_{12} J_1\left(\frac{7.01r}{a}\right)\cos(\phi + \phi_{12})\cos\left(\frac{7.01}{a}ct + \Omega_{12}\right)$$

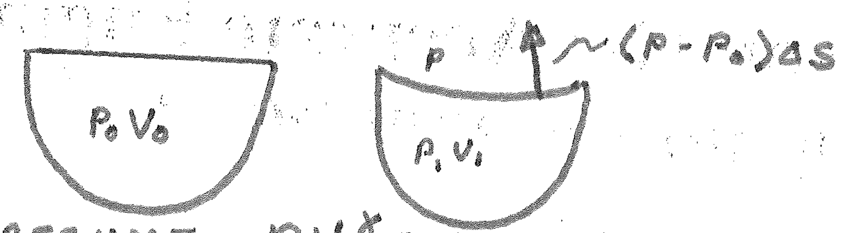
$$\vdots$$

$$Z_{21} = C_{21} J_2\left(\frac{5.15r}{a}\right)\cos(2\phi + \phi_{21})\cos\left(\frac{5.15}{a}ct + \Omega_{21}\right)$$

$$\downarrow$$

D) KETTLE DRUM

1) DERIVATION



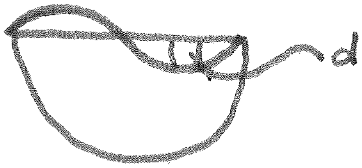
ASSUME $PV^{\gamma} = $ CONSTANT

[TENSILE FORCES] + $[P-P_0]\Delta S = \sigma \Delta S \frac{\delta^2 z}{\delta t^2}$

$$c^2\left[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r} + \frac{1}{r^2}\frac{\delta^2 z}{\delta \phi^2}\right] + \frac{P-P_0}{\sigma} = \frac{\delta^2 z}{\delta t^2}$$

$\Delta P = dP = \frac{-\gamma P_0}{V_0} dV$ (PREVIOUSLY DERIVED)

$$c^2\left[\frac{\delta^2 z}{\delta r^2} + \frac{1}{r}\frac{\delta z}{\delta r} + \frac{1}{r^2}\frac{\delta^2 z}{\delta \phi^2}\right] - \frac{\gamma P_0}{V_0} dV = \sigma \Delta S \frac{\delta^2 z}{\delta t^2}$$



$$dV = \int_0^{2\pi}\int_0^{a} z(r,\phi,t)\, r\, dr\, d\phi$$

2) SOLUTION

$$z(r,\phi,t) = \psi(r,\phi) H(t)$$

$$V_0 = \int_0^{2\pi}\int_0^{a} \psi(r,\phi) H(t)\, r\, dr\, d\phi = H I_0$$

$$\Rightarrow c^2\left[H\frac{\delta^2 \psi}{\delta r^2} + \frac{H}{r}\frac{\delta \psi}{\delta r} + \frac{H}{r^2}\frac{\delta^2 \psi}{\delta \phi^2}\right] - \frac{\gamma P_0}{V_0 \sigma} I_0 H = \psi \frac{\delta^2 H}{\delta t^2}$$

$$\frac{c^2}{\psi}\left[\frac{\delta^2 \psi}{\delta r^2} + \frac{1}{r}\frac{\delta \psi}{\delta r} + \frac{1}{r^2}\frac{\delta^2 \psi}{\delta \phi^2}\right] - \frac{\gamma P_0 I_0}{V_0 \sigma \psi} = \frac{1}{H}\frac{\delta^2 H}{\delta t^2} = -\omega^2$$

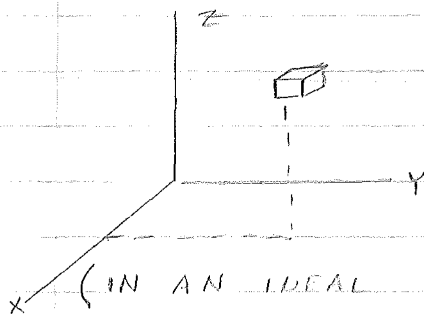
$$\Rightarrow H = d_1 \cos \omega t + d_2 \sin \omega t$$

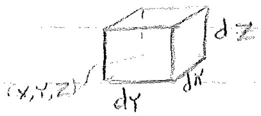
$$\frac{\delta^2 \psi}{\delta r^2} + \frac{1}{r}\frac{\delta \psi}{\delta r} + \frac{1}{r^2}\frac{\delta^2 \psi}{\delta \phi^2} + K^2 \psi = \frac{\gamma P_0 I_0}{V_0 \sigma c^2}$$

$$\psi(r,\phi) = R(r) + \Phi(\phi)$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \frac{1}{r^2}\frac{d^2 \Phi}{\delta \phi^2} + K^2 = 0$$

WAVES IN FLUIDS





DISPLACEMENTS ξ, η, ζ

$\Rightarrow$ STRAINS $\frac{\delta \xi}{\delta x}, \frac{\delta \eta}{\delta y}, \frac{\delta \zeta}{\delta z}$

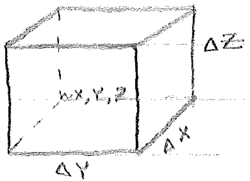
(IN AN IDEAL FLUID, THERE ARE NO SHEARING STRAINS)

THUS, THE STRESS STRAIN RELATIONSHIP FOR A FLUID IS

$$\Delta P = -B\left[\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta y} + \frac{\delta \zeta}{\delta z}\right] \quad Ⓐ$$

$$= -B\left[\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right]$$

THE WAVE EQUATION:



ON THE FRONT AND BACK FACES, FROM NEWTON'S SECOND LAW:

$$P'(x,y,z)\,\Delta y \Delta z - P'(x+\Delta x, y, z)\,\Delta y\,\Delta z = \rho\,\Delta x \Delta y \Delta z \frac{\delta^2 \xi}{\delta t^2}$$

OR $-\frac{\delta P'}{\delta x} = \rho \frac{\delta^2 \xi}{\delta t^2}$

AND SIMILARLY: $-\frac{\delta P'}{\delta y} = \rho \frac{\delta^2 \eta}{\delta t^2}$

AND: $-\frac{\delta P'}{\delta z} = \rho \frac{\delta^2 \zeta}{\delta t^2}$

COMBINING:

$$-\left(\frac{\delta P'}{\delta x} + \frac{\delta P'}{\delta y} + \frac{\delta P'}{\delta z}\right) = \rho\left(\frac{\delta^2 \xi}{\delta t^2} + \frac{\delta^2 \eta}{\delta t^2} + \frac{\delta^2 \zeta}{\delta t^2}\right)$$

$$-\text{grad } P' = \rho \frac{\delta^2 \vec{s}}{\delta t^2} = \frac{\delta \vec{v}}{\delta t^2} \quad \Rightarrow \vec{v} = \delta \vec{s}/\delta t$$

RECALL THE STRESS STRAIN RELATIONSHIP Ⓐ

$$P = -B\left[\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta y} + \frac{\delta \zeta}{\delta z}\right]$$

$$\frac{\delta^2 P}{\delta t^2} = -B\left[\frac{\delta}{\delta x}\frac{\delta^2 \xi}{\delta t^2} + \frac{\delta}{\delta y}\frac{\delta^2 \eta}{\delta t^2} + \frac{\delta}{\delta z}\frac{\delta^2 \zeta}{\delta t^2}\right]$$

$$= -B\left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right]\left[\frac{\delta^2 \xi}{\delta t^2} + \frac{\delta^2 \eta}{\delta t^2} + \frac{\delta^2 \zeta}{\delta t^2}\right]$$

$$= -B\left[\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right]\left[-\frac{1}{\rho}\right]\left[\frac{\delta P}{\delta x} + \frac{\delta P}{\delta y} + \frac{\delta P}{\delta z}\right]$$

$$= \frac{B}{\rho}\left[\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta y^2} + \frac{\delta^2 P}{\delta z^2}\right]$$

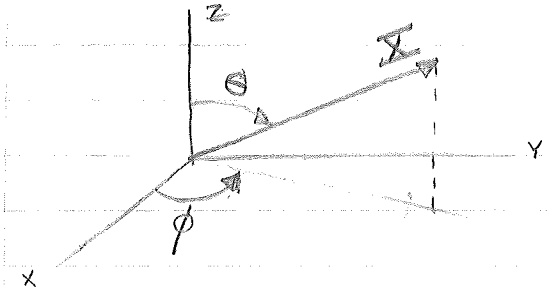
$$\frac{B}{\rho} = c^2$$

SOLUTION OF THE WAVE EQUATION:

A) $P = Ae^{i(\omega t - kx)}$ ⇒ A PLANE WAVE

OR $f(v) \Rightarrow v = ct - (x \sin\theta\cos\phi + Y \cos\theta \sin\phi + z \cos\theta)$

$= ct \cdot \mathbb{X}$



FOR AN IDEAL GAS:

$$B_a = \gamma P \Rightarrow C = \sqrt{\frac{\gamma P}{\rho}}$$

$$PV = nRT \Rightarrow V = \frac{nRT}{V} \Rightarrow \frac{P}{\rho} = \frac{RT}{m}$$

$$c = \sqrt{\gamma RT/m} = \text{CONST.}\sqrt{T}$$

B) HARMONIC SOLUTION

AGAIN $c^2\left[\frac{\delta^2 P}{\delta X^2} + \frac{\delta^2 P}{\delta Y^2} + \frac{\delta^2 P}{\delta Z^2}\right] = \frac{\delta^2 P}{\delta t^2}$

LET $P(X, Y, Z, t) = \mathbb{X}(x)\,\mathbb{Y}(y)\,\mathbb{Z}(z)\,H(t)$

$$\Rightarrow c^2\left[\frac{1}{\mathbb{X}}\frac{d^2\mathbb{X}}{dX^2} + \frac{1}{\mathbb{Y}}\frac{d^2\mathbb{Y}}{dY^2} + \frac{1}{\mathbb{Z}}\frac{d^2\mathbb{Z}}{\delta Z^2}\right] = \frac{1}{H}\frac{d^2H}{dt^2} = -\omega^2$$

$$\frac{d^2H}{dt^2} = -\omega^2 H$$

$$\frac{1}{\mathbb{X}}\frac{d^2\mathbb{X}}{dX^2} = -\frac{1}{\mathbb{Y}}\frac{d^2\mathbb{Y}}{dY^2} - \frac{1}{\mathbb{Z}}\frac{d^2\mathbb{Z}}{dZ^2} - k^2 = -\alpha^2$$

$$\frac{d^2\mathbb{X}}{dX^2} = -\alpha^2\mathbb{X}$$

$$+\frac{1}{Y}\frac{d^2Y}{dY^2} = -\alpha^2 + k^2 + \frac{1}{\mathbb{Z}}\frac{d^2\mathbb{Z}}{dZ^2} = -B^2$$

$$\frac{d^2Y}{dY^2} = -B^2\mathbb{Y}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -(k^2 - \alpha^2 - B^2) = \gamma^2$$

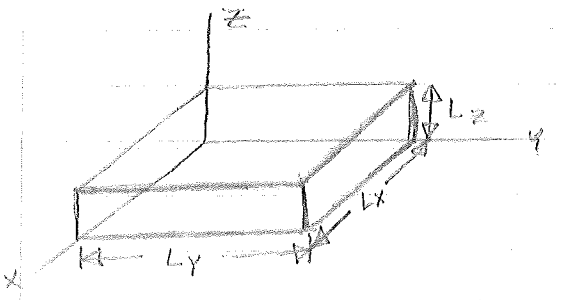
$$\frac{d^2Z}{dZ^2} = -\gamma^2 Z$$

THUS $P(X, Y, Z, t) = [a_1 \cos\alpha X + b_1 \sin\alpha X]$
$[a_2 \cos BY + b_2 \sin BY]$
$[a_3 \cos\gamma Z + b_3 \sin\gamma Z]$
$[a_4 \cos\omega t + b_4 \sin\omega t]$

$= A\cos(\alpha X + \Omega_1)\cos(BY + \Omega_2)$
$\cos(\gamma X + \Omega_3)\cos(\omega t + \Omega_4)$

$P(X, Y, Z, t)$ MAY BE EXPRESSED AS 8 PLANE WAVES

BOUNDRY CONDITIONS:



$$\xi(0,Y,z,t) = \frac{\delta \xi}{\delta t}\Big|_{0Yzt} = \frac{\delta^2 \xi}{\delta t^2}\Big|_{0Yzt} = 0 \quad \text{SINCE } -\frac{\delta P}{\delta X} = \rho \frac{\delta^2 \xi}{\delta t^2};$$

$$\Rightarrow \frac{\delta P}{\delta X}\Big|_{0,Y,z,t} = 0 \Longrightarrow b_1 = 0$$

$$\eta(x,0,z,t) = \frac{\delta \eta}{\delta t}\Big|_{x,0,zt} = \frac{\delta^2 \eta}{\delta t^2}\Big|_{x,0,zt} = 0 \quad \text{SINCE } -\frac{\delta P}{d Y} = \rho \frac{\delta^2 \eta}{\delta t^2}$$

$$\Rightarrow \frac{\delta P}{\delta Y}\Big|_{x,0,z,t} = 0 \Longrightarrow b_2 = 0$$

$$\zeta(x,Y,0,t) = \frac{\delta \zeta}{\delta t}\Big|_{x,Y,0,t} = \frac{\delta^2 \zeta}{\delta t^2}\Big|_{x,Y,0,t} \quad \text{SINCE } -\frac{\delta P}{\delta z} = \rho \frac{\delta^2 \zeta}{\delta t^2}$$

$$\Rightarrow \frac{\delta P}{\delta z}\Big|_{x,Y,0,t} = 0 \Longrightarrow b_3 = 0$$

REDUCING THE EQUATION:

$$P(x,Y,z,t) = (a_1 a_2 a_3) \cos \alpha x \cos \beta Y \cos \gamma z \left[a_4 \cos \omega t + b_4 \sin \omega t\right]$$

$$0 = \xi(L_x, Y, z, t) \Rightarrow \frac{\delta P}{\delta X}\Big|_{L_x,Y,z,t} = 0 \Rightarrow \alpha = \frac{n_x \pi}{L_x}$$

$$0 = \eta(X, L_Y, Z, t) \Rightarrow \frac{\delta P}{\delta Y}\Big|_{X,L_Y,Z,t} = 0 \Rightarrow \beta = \frac{n_Y \pi}{L_Y}$$

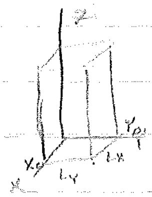
$$0 = \zeta(X, Y, L_z, t) \Rightarrow \frac{\delta P}{\delta z}\Big|_{X,Y,L_z,t} = 0 \Rightarrow \gamma = \frac{n_z \pi}{L_z}$$

$$\text{NOW} \quad K^2 = \alpha^2 + \beta^2 + \gamma^2 = \left(\frac{\omega}{c}\right)^2$$

$$\Rightarrow \omega_{n_x n_Y n_z} = \pi c \sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_Y}{L_Y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2}$$

$$\text{AND: } P_{n_x, n_Y, n_z} = \cos \frac{n_x \pi}{L_x} X \cos \frac{n_Y \pi}{L_Y} Y \cos \frac{n_z \pi}{L_z} z \left[A_{n_x n_Y n_z} \cos \omega_{n_x n_Y n_z} t + B_{n_x n_Y n_z} \sin \omega_{n_x n_Y n_z} t\right]$$

THE WAVEGUIDE



$$P(x,Y,z,t) = A\cos(\alpha x + \Omega_1)\cos(\beta Y + \Omega_2)$$
$$\cos(\gamma z + \Omega_3)\cos(\omega t + \Omega_4)$$

$$P_{0,Y,z,t} = P_{L_x,Y,z,t} = 0$$
$$P_{x,0,z,t} = P_{x,L_Y,z,t} = 0$$

Ⓐ
$$\rightarrow P(x,Y,z,t) = C\cos\frac{n_y\pi}{L_x}X\cos\frac{n_y\pi}{L_Y}Y\cos(\gamma z+\delta)\cos(\omega t+\Omega)$$
$$P(x_0,Y_0,z,t) = A'\cos(\gamma z+\delta)\cos(\omega t+\Omega)$$
$$= \frac{A}{2}\left[\cos(\gamma z + \omega t + \Omega) + \cos(\gamma z + \delta - \omega t - \Omega)\right]$$
$$= \frac{A}{2}\left[\cos\gamma\left(z + \frac{\omega}{\gamma}t + \frac{\Omega+\delta}{\gamma}\right) + \cos\gamma\left(z - \frac{\omega}{\gamma}t + \frac{\delta-\Omega}{\gamma}\right)\right]$$

NOTE $c' = \frac{\omega}{\gamma}$

$$= \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n_x\pi}{L_x}\right)^2 - \left(\frac{n_y\pi}{L_Y}\right)^2}}$$

$\exists$ A SOLUTION FOR Ⓐ WHEN

$$\left(\frac{\omega}{c}\right)^2 > \left(\frac{n_x\pi}{L_x}\right)^2 + \left(\frac{n_y\pi}{L_Y}\right)^2$$

THE (0,0) MODE MAY BE INSURED IF

$$\left(\frac{\omega}{c}\right)^2 < \left(\frac{\pi}{L_x}\right)^2 + \left(\frac{\pi}{L_Y}\right)^2$$

PLANE WAVES

THE WAVE EQUATION:

$$c^2\left[\frac{\delta^2 P}{\delta X^2} + \frac{\delta^2 P}{\delta Y^2} + \frac{\delta^2 P}{\delta Z^2}\right] = \frac{\delta^2 P}{\delta t^2}$$

YIELDS A PLANE WAVE SOLUTION:

$$P(x,Y) = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)} \quad \ni k = \frac{\omega}{c}$$

NEWTON'S SECOND LAW:

$$-\frac{\delta P}{\delta x} = \rho \frac{\delta^2 \xi}{\delta x^2}$$

$$\Rightarrow U = \frac{\delta \xi}{\delta t} = \frac{A}{\rho c} e^{i(\omega t - kx)} - \frac{B}{\rho c} e^{i(\omega t + kx)}$$

DEFINE THE SPECIFIC ACOUSIC IMPEDANCE:

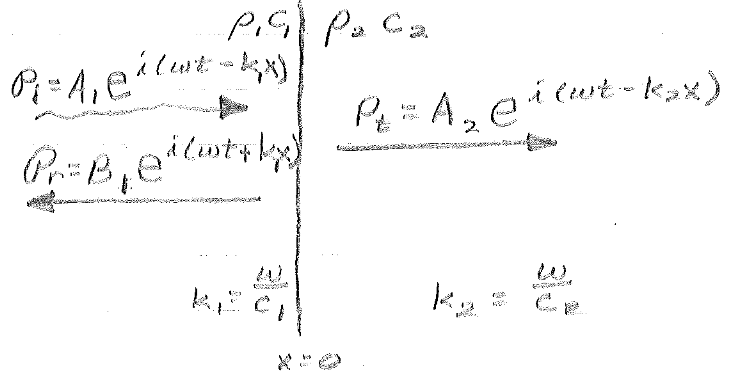
$$\triangleq P/U$$

$$\frac{P_x}{U_x} = \frac{Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}}{\frac{A}{\rho c}e^{i(\omega t - kx)} - \frac{B}{\rho c}e^{i(\omega t + kx)}}$$

$$= \rho c \frac{Ae^{ikx} + Be^{-ikx}}{Ae^{ikx} + Be^{-ikx}}$$

WHERE $\rho c \triangleq$ CHARACTERISTIC IMPEDANCE

CONSIDER TWO MEDIA



$$P_{LEFT} = A_1 e^{i(\omega t - k_1 x)} + B_1 e^{i(\omega t + k_1 x)}$$

$$P_{RIGHT} = A_2 e^{i(\omega t - k_2 x)}$$

$$U_{LEFT} = \frac{1}{\rho_1 c_1}\left[A_1 e^{i(\omega t - k_1 x)} - B_1 e^{i(\omega t + k_1 x)}\right]$$

$$U_{RIGHT} = \frac{1}{\rho_2 c_2}\left[A_2 e^{i(\omega t - k_2 x)}\right]$$

BOUNDRY CONDITIONS:

$$P_L\big|_{x=0} = P_R\big|_{x=0} \Rightarrow A_1 + B_1 = A_2$$

$$U_L\big|_{x=0} = U_R\big|_{x=0} \Rightarrow \frac{A_1 - B_1}{\rho_1 c_1} = \frac{A_2}{\rho_2 c_2}$$

THUS $\frac{A_1 + B_1}{A_1 - B_1} = \frac{\rho_2 c_2}{\rho_1 c_1}$

OR $$\frac{B_1}{A_1} = \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1}$$

AND $$\frac{A_2}{A_1} = \frac{2\rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}$$

THEN: $$P_i = A_1 e^{i(\omega t - kx)}$$

$$P_r = A_1 \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1} e^{i(\omega t + kx)}$$

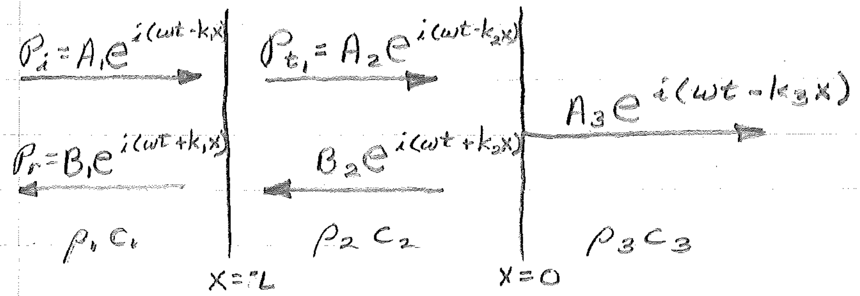
$$P_t = \frac{2\rho_2 c_2 A_1}{\rho_1 c_1 + \rho_2 c_2} e^{i(\omega t - kx)}$$

AND @ $x=0$ (AT THE BOUNDRY)

$$P_i = A_1 e^{i\omega t}$$
$$P_r = A_1 \frac{\rho_2 c_2/\rho_1 c_1 - 1}{\rho_2 c_2/\rho_1 c_1 + 1} e^{i\omega t}$$
$$P_t = A_1 \frac{2\rho_2 c_2}{\rho_2 c_2 + \rho_1 c_1} e^{i\omega t}$$

NOTE THAT P_i AND P_t ARE IN PHASE. P_r & P_i ARE IN PHASE OR OUT OF PHASE, DEPENDING ON THE SIGN OF $\rho_2 c_2/\rho_1 c_1 - 1$

CONSIDER 3 MEDIA (STEADY STATE)



$$P_{MIDDLE}\big|_{x=0} = P_{RIGHT}\big|_{x=0} \Rightarrow A_2 + B_2 = A_3$$

$$U_{MIDDLE}\big|_{x=0} = U_{RIGHT}\big|_{x=0} \Rightarrow \frac{1}{\rho_2 c_2}(A_2 - B_2) = \frac{1}{\rho_3 c_3} A_3$$

EXPRESSIONS SIMILAR TO THE TWO MEDIA CASE.

$$\Rightarrow \frac{B_2}{A_2} = \frac{\frac{\rho_3 c_3}{\rho_2 c_2} - 1}{\frac{\rho_3 c_3}{\rho_2 c_2} + 1} = \frac{r_{23} - 1}{r_{23} + 1}$$

$$\frac{A_3}{A_2} = \frac{2\rho_3 c_3}{\rho_3 c_3 + \rho_2 c_2}$$

BOUNDRY CONDITIONS @ $X = -L$

$$P_{LEFT}\big|_{x=-L} = P_{MIDDLE}\big|_{x=-L}$$

$$\Rightarrow A_1 e^{+ik_1 L} + B_1 e^{-ik_1 L} = A_2 e^{ik_2 L} + B_2 e^{-ik_2 L}$$

$$U_{LEFT}\big|_{x=-L} = U_{RIGHT}\big|_{x=-L}$$

$$\frac{1}{\rho_1 c_1}\left[A_1 e^{+ik_1 L} - B_1 e^{-ik_1 L}\right] = \frac{1}{\rho_2 c_2}\left[A_2 e^{ik_2 L} - B_2 e^{-ik_2 L}\right]$$

$$\Rightarrow A_1 e^{ik_1 L} - B_1 e^{-ik_1 L} = r_{21}\left[A_2 e^{ik_2 L} - B_2 e^{-ik_2 L}\right]$$

$$\frac{A_1 e^{ik_1 L} + B_1 e^{-ik_1 L}}{A_2 e^{ik_2 L} - B_2 e^{-ik_2 L}} = r_{21} \frac{e^{ik_1 L} + \frac{B_2}{A_1} e^{-ik_2 L}}{e^{ik_2 L} - \frac{B_2}{A_2} e^{-ik_2 L}}$$

$$= r_{21} \frac{e^{ik_2 L} + \frac{r_{23} - 1}{r_{23} + 1} e^{-ik_2 L}}{e^{ik_2 L} - \frac{r_{23} - 1}{r_{23} + 1} e^{-ik_2 L}}$$

$$= r_{23} \frac{2 r_{23} \cos k_2 L + i\, 2 \sin k_2 L}{2 i\, r_{23} \cos k_2 L + 2 \cos k_2 L}$$

THEN

$$\frac{B_1 e^{-ik_1L}}{A_1 e^{ik_1L}} = \frac{(r_{12}r_{23}-1)\cos k_2L + i(r_{12}-r_{23})\sin k_2L}{(r_{12}r_{23}+1)\cos k_2L + i(r_{12}+r_{23})\sin k_2L}$$

$$\frac{B_1}{A_1} = \left[\frac{(r_{13}-1)\cos k_2L + i(r_{12}-r_{23})\sin k_2L}{(r_{13}+1)\cos k_2L + i(r_{12}+r_{23})\sin k_2L}\right] e^{i2k_1L}$$

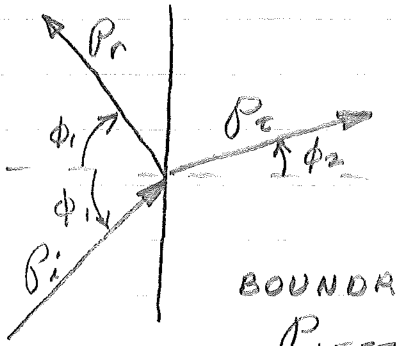
CHOOSE $k_2L = \frac{(2n+1)\pi}{2}$ $\quad n = 0, 1, 2, \ldots$

$r_{12} = r_{23}$ $\quad (\rho_2 c_2 = \sqrt{\rho_1 c_1 \, \rho_3 c_3})$

THEN $B_1 = 0$ (NO REFLECTION)

NON NORMAL RAYS:

$$P_i = A_1 e^{i[\omega t - k(x\cos\phi_1 + y\sin\phi_1)]}$$



$$P_r = B_1 e^{i[\omega t - k_1(-x\cos\phi_1 + y\sin\phi_1)]}$$
$$P_t = A_2 e^{i[\omega t - k_2(x\cos\phi_2 + y\sin\phi_2)]}$$
$$U_i = \frac{A_1}{\rho_1 c_1} e^{i[\omega t - k_1(x\cos\phi_1 + y\sin\phi_1)]}$$
$$U_R = \frac{B_2}{\rho_1 c_1} e^{i[\omega t - k_1(-x\cos\phi_1 + y\sin\phi_1)]}$$
$$U_t = \frac{A_2}{\rho_2 c_2} e^{i[\omega t - k_2(+x\cos\phi_2 + y\sin\phi_2)]}$$

BOUNDRY CONDITIONS:

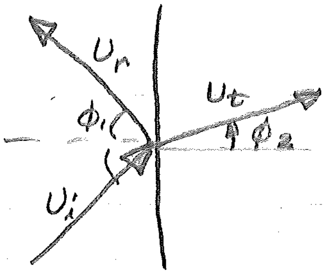
$$P_{LEFT}\Big|_{x=0} = P_{RIGHT}\Big|_{x=0}$$

$$\Rightarrow A_1 e^{i[\omega t - k_1 y\sin\phi_1]} + B_1 e^{i[\omega t - k_1 \sin\phi_1]} = A_2 e^{i[\omega t - k_2 \sin\phi_2]}$$

BUT $k_1 \sin\phi_1 = k_2 \sin\phi_2$

$\Rightarrow A_1 + B_1 = A_2$

VELOCITY BOUNDRY CONDITIONS:



$$[U_1\cos\phi_1 - U_r\cos\phi_1]_{x=0} = U_t\cos\phi_2\Big|_{x=0}$$

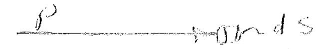
$$\left[\frac{A_1}{\rho_1 c_1} - \frac{B_1}{\rho_1 c_1}\right]\cos\phi_1 = \frac{A_2}{\rho_2 c_2}\cos\phi_2$$

$$\frac{A_1 - B_1}{A_1 + B_1} = \frac{\rho_2 c_2}{\rho_1 c_1}\,\frac{\cos\phi_2}{\cos\phi_1}$$

$$\frac{B_1}{A_1} = \frac{\frac{\rho_2 c_2}{\rho_1 c_1}\,\frac{\cos\phi_2}{\cos\phi_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1}\,\frac{\cos\phi_2}{\cos\phi} + 1}$$

FOR NO RELECTION: $\frac{\rho_2 c_2}{\rho_1 c_1}\,\frac{\cos\phi_2}{\cos\phi_1} = 1 \Rightarrow \cos\phi_1 = \frac{\rho_2 c_2}{\rho_1 c_1}\cos\phi_2$

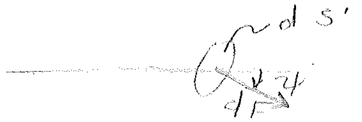
ENERGY CONSIDERATIONS



$\rho = Ae^{i(\omega t - kx)}$

$dF = P ds$

$\frac{dE}{\delta t} = P ds\, U = dF\, v$



$$\frac{\delta E}{\delta t} = P ds' U \cos\psi$$

$$\left(\frac{\delta E}{\delta t}\right)_{AVE} = \frac{1}{T}\int_0^T P ds\, U \cos\psi\, dt$$

$$= \frac{1}{T}\int_0^T \underset{\uparrow}{A}\cos(\omega t - kx + \alpha)\frac{A}{\rho c}\cos(\omega t - kx + \alpha)\, dt\, ds$$

FOR $\psi = 0$

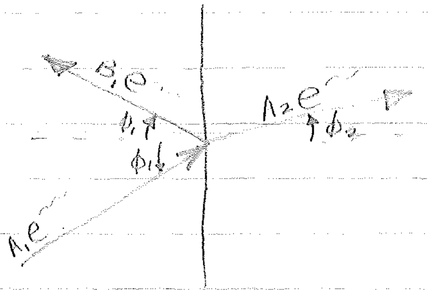
$$\left(\frac{\delta E}{\delta t}\right)_{AVE} = \frac{A^2 ds}{\rho c T}\int_0^T \cos^2(\omega t - kx + \alpha)\, dt$$

$$= \frac{A^2}{2\rho c} ds$$

I = INTENSITY = $\frac{\text{RATE}}{\text{UNIT AREA}}$ @ WHICE ENERGY CROSSES A SMALL AREA PERPENDICULAR TO DIRECTION OF PROPAGATION

$$I = \frac{A^2}{2\rho c}$$

ENERGY BOUNDRY CONDITIONS



$$\frac{|A_1|^2}{2\rho_1 c_1} ds \cos\phi_1 - \frac{|B_1|^2}{2\rho_1 c_1} ds \cos\phi_1 = \frac{|A_2|^2}{2\rho_2 c_2} ds \cos\phi_2$$

$$\Rightarrow \frac{|A_2|^2}{|A_1|^2} = \frac{\rho_2 c_2}{\rho_1 c_1}\left[1 - \frac{|B_1|^2}{|A_1|^2}\right]\frac{\cos\phi_2}{\cos\phi_1}$$

RECALL $\frac{|B_1|}{|A_1|} = \frac{r_{12} - 1}{r_{12} + 1}$

Chapter I Problems

1.1 When an object undergoes a change in volume due to applied stresses the quantity $\Delta V/V$ is defined as the volume strain or dilation. Show, for a rod of cross-sectional area A, subjected to equal and opposite forces of magnitude F at its two ends, that

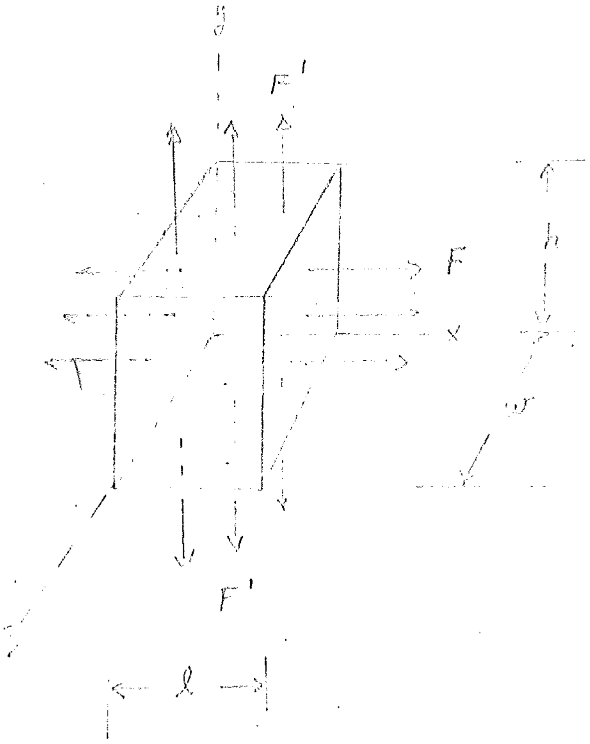
$$\frac{\Delta V}{V} = \frac{F}{AY}(1 - 2\sigma)$$

1.2 A block of dimensions, ℓ, w and h is subjected to forces on four of its six faces as indicated in the accompanying figure. If the height, h, remains unchanged when the forces are applied, show that

$$\frac{S_{xx}}{e_{xx}} = \frac{Y}{1 - \sigma^2}$$

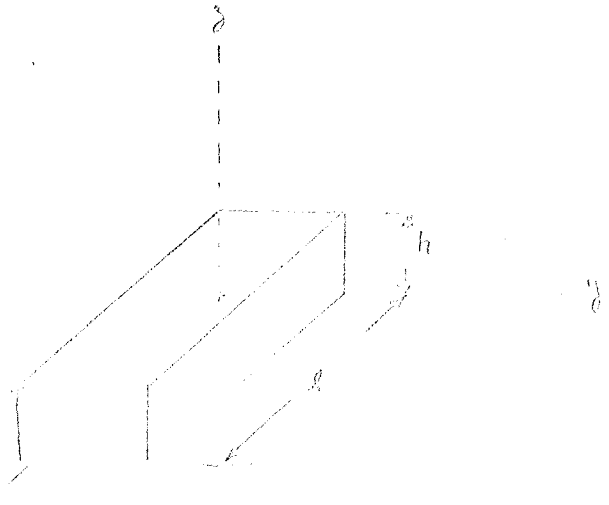
and

$$\frac{e_{zz}}{e_{xx}} = - \frac{\sigma}{1 - \sigma}$$



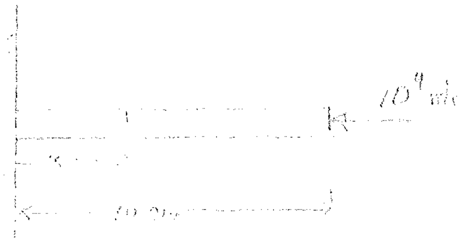
1.3 A block of dimensions ℓ, w, h is subjected to forces on all six faces, the forces being of such magnitude that the dimensions w and h remain unchanged when the forces are applied. Show that

$$S_{yy} = S_{zz} = \frac{\sigma}{1-\sigma} S_{xx}$$



1.4 Solids and liquids are only slightly compressible and the bulk modulus $B = -\Delta P/(\Delta V/V)$ is essentially independent of the size of ΔP and the mean pressure at which the measurement is made. This is not true for gases; it is only for very small changes of pressure about some mean pressure for which the quantity $\Delta P/(\Delta V/V)$ is a constant. The equation of state of an ideal gas is PV = nRT where n is the number of moles of the gas and R is the gas constant. Show that for <u>small</u> changes about some equilibrium state characterized by P_0, V_0, the <u>isothermal</u> bulk modulus is equal to P_0. When an ideal gas undergoes an adiabatic process, the quantity PV^{γ} remains constant (γ is the ratio of the specific heat of the gas at constant pressure to that at constant volume.) Show that for small changes about some equilibrium state characterized by P_0, V_0, the adiabatic bulk modulus is γP_0.

1.5. A brass rod 50 cm long and of square cross-section of 1 cm^2 area is compressed against a rigid wall by a force of 10^4 nts as indicated in the sketch below. Find the stress component S_{xx} at a point P, a distance x from the wall. Find ϵ_{xx}, ϵ_{yy}, and ϵ_{zz} at P. Find the displacement ξ of a cross-section 30 cm from the wall.



1.6 When a uniform rod is suspended from one end under its own weight the strain component $\epsilon_{xx} = \frac{1}{Y}\rho g(\ell_0 - x)$ where ρ is the density and ℓ_0 the unstretched length. Each small piece of length dx in the unstressed rod is stretched an amount $d\xi = \epsilon_{xx}\,dx$. Find how ϵ_{yy} varies with x, and find the length λ of the stressed rod in terms of ℓ_0, Y, ρ and g.

1.7 Which of the equations (1.13), (1.14), (1.15) and (1.16) are correct for all values of x from x = 0 to x = L. Which need to be modified for $x > \frac{L}{2}$?

1.8 A light beam of circular cross-section of radius a is supported on two knife edges at its ends and loaded in the center by a weight W. Show that the bending moment at a point is given by

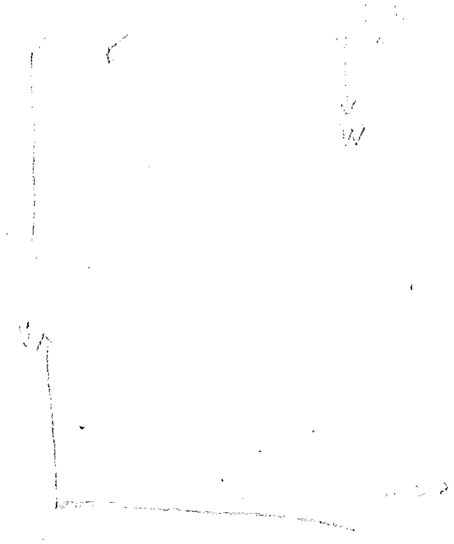
$$M = \frac{Y \pi a^4}{4} \frac{d^2 y}{dx^2}$$

where y(x) is the equation of the center line of the distorted beam.

1.9 One end of a light beam is clamped in a wall and a load W is hung from the other end.

(a) Assuming the forces exerted by the wall on the beam can be represented by a single force F_0 and a couple of moment M_0, find M_0 and the components of F_0 by isolating the entire beam.

(b) If the dimensions of the beam are L, w and $\overset{h}{H}$ and the distortion undergone by the beam is small, find the bending moment as a function of x and determine the equation y(x) of the bent beam.



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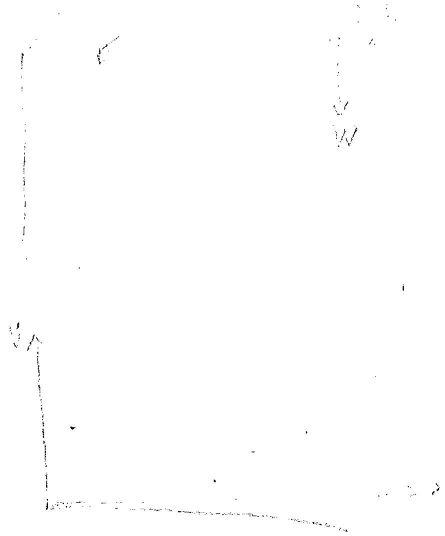
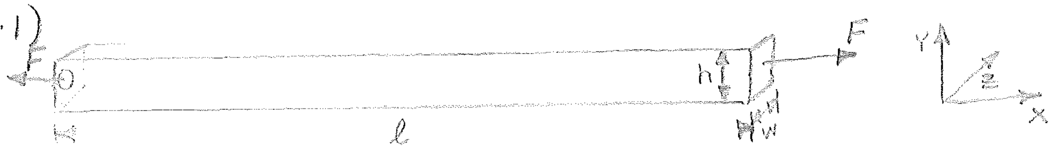


(b) If the dimensions of the beam are L, w and $\overset{h}{H}$ and the distortion undergone by the beam is small, find the bending moment as a function of x and determine the equation y(x) of the bent beam.

BOB MARKS

$\frac{5.9}{6}$

(1-1)



$S_{YY} = S_{ZZ} = 0$

$$\Rightarrow \begin{bmatrix} \epsilon_{XX} \\ \epsilon_{YY} \\ \epsilon_{ZZ} \end{bmatrix} = \frac{F}{AY}\begin{bmatrix} 1 \\ -\sigma \\ -\sigma \end{bmatrix} = \begin{bmatrix} \epsilon_{XX} \\ -\sigma\epsilon_{XX} \\ -\sigma\epsilon_{XX} \end{bmatrix} = \begin{bmatrix} \frac{\ell'-\ell}{\ell} \\ \frac{h'-h}{h} \\ \frac{w'-w}{w} \end{bmatrix}$$

LET $V' = \ell' w' h'$ AND $\Delta V = V' - V = \ell' w' h' - \ell w h$

WHERE PRIMED VARIABLES ARE SYSTEM PARAMETERS WHEN FORCE F IS APLIED

FROM ABOVE: $\begin{bmatrix} \ell' \\ h' \\ w' \end{bmatrix} = \begin{bmatrix} \ell(1+\epsilon_{XX}) \\ h(1-\sigma\epsilon_{XX}) \\ w(1-\sigma\epsilon_{XX}) \end{bmatrix}$

THUS: $V' = \ell' h' w' = \ell h w (1+\epsilon_{XX})(1-\sigma\epsilon_{XX})^2 = V(1+\epsilon_{XX})(1-\sigma\epsilon_{XX})^2$

AND: $\frac{\Delta V}{V} = \frac{V'-V}{V} = (1+\epsilon_{XX})(1-\sigma\epsilon_{XX})^2 - 1$

EXPANDING:

$$\frac{\Delta V}{V} = \epsilon_{XX} - 2\sigma\epsilon_{XX}^2 + \sigma^2\epsilon_{XX}^3 + 1 - 2\sigma\epsilon_{XX} + \sigma^2\epsilon_{XX}^2 - 1$$

IN THAT $0 < \epsilon_{XX} \ll 1$, $0 < \epsilon_{XX}^3 \ll \epsilon_{XX}^2 \lll \epsilon_{XX} \ll 1$.

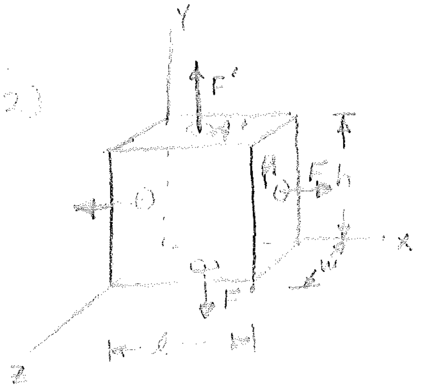
THUS, THE HIGHER ORDER TERMS MAY BE DROPPED, (RECOGNIZING $\sigma < 1/2$).

THEN: $\frac{\Delta V}{V} \cong \epsilon_{XX} - 2\sigma\epsilon_{XX}$

$= \epsilon_{XX}(1-2\sigma)$

$= \frac{F}{YA}(1-2\sigma)$ ✓

(1-2)



$$S_{zz} = 0 \; ; \; \epsilon_{YY} = \frac{h' - h}{h} = 0$$

$$\begin{bmatrix} \epsilon_{xx} \\ 0 \\ \epsilon_{zz} \end{bmatrix} = \frac{1}{Y} \begin{bmatrix} 1 & -\sigma & -\sigma \\ -\sigma & 1 & -\sigma \\ -\sigma & -\sigma & 1 \end{bmatrix} \begin{bmatrix} S_{xx} \\ S_{YY} \\ 0 \end{bmatrix}$$

WHERE $S_{xx} = F/A$, AND $S_{YY} = F'/A'$

a) $\epsilon_{xx} = \frac{1}{Y}\left[S_{xx} - \sigma S_{YY}\right]$ AND $0 = -\sigma S_{xx} + S_{YY}$ ✓

COMBINING:

$$\epsilon_{xx} = \frac{1}{Y}\left[S_{xx} - \sigma^2 S_{xx}\right]$$

$$= \frac{S_{xx}}{Y}\left[1 - \sigma^2\right] \quad ✓$$

$$\Rightarrow \frac{S_{xx}}{\epsilon_{xx}} = \frac{Y}{1 - \sigma^2}$$

b) $\epsilon_{xx} = \frac{S_{xx}}{Y}\left[1 - \sigma^2\right]$ (FROM ABOVE)

$$\epsilon_{zz} = -\frac{\sigma}{Y}(S_{xx} + S_{YY})$$

$$= -\frac{\sigma}{Y}(S_{xx} + \sigma S_{xx})$$

$$= -\frac{\sigma}{Y} S_{xx}(1 + \sigma)$$

THUS:

$$\frac{\epsilon_{zz}}{\epsilon_{xx}} = \frac{-\frac{\sigma}{Y} S_{xx}(1 + \sigma)}{\frac{1}{Y} S_{xx}(1 - \sigma^2)}$$

$$= \frac{-\sigma(1 + \sigma)}{(1 - \sigma^2)}$$

$$= \frac{-\sigma(1 + \sigma)}{(1 + \sigma)(1 - \sigma)}$$

$$= \frac{-\sigma}{1 - \sigma}$$

✓

(1-4) $B = \frac{-\Delta P}{\Delta V/V} = -\left(\frac{\Delta P}{\Delta V}\right) V$

FOR SMALL CHANGES;

$$B = -\frac{dP}{dV} V$$

a) ISOTHERMAL

$$P_0 V_0 = nRT_0$$

$$P = \frac{nRT_0}{V_0}$$

$$\frac{\Delta P}{\Delta V} \simeq \left.\frac{\partial P}{\partial V}\right|_{T_0} - \frac{nRT_0}{V_0^2}$$

$$B = \left(\frac{nRT_0}{V_0^2}\right) V_0$$

$$= \frac{nRT}{V} = P_0 \quad ✓$$

b) ADIABATIC

LET $P_0 V_0^{\gamma} \cdot C$ ∋ C IS A CONSTANT

$$\Rightarrow P_0 = C V_0^{-\gamma}$$

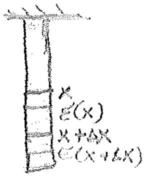
$$\frac{\Delta P}{\Delta V} \simeq \frac{dP}{dV} = -\gamma C V_0^{-(\gamma+1)}$$

$$B = -\left[-\gamma C V_0^{-(\gamma+1)}\right] V_0$$

$$= +\gamma C V_0^{-\gamma}$$

$$= +\gamma P_0 \quad ✓$$

(1-6)



$$\epsilon_{xx} = \frac{1}{Y}\rho g(\ell_0 - x)$$
$$d\xi = \epsilon_{xx}dx$$

a) FROM (1-4)

$$\epsilon_{xx} = \frac{1}{Y}S_{xx} - \frac{\sigma}{Y}S_{yy} - \frac{\sigma}{Y}S_{zz}$$
$$\epsilon_{yy} = -\frac{\sigma}{Y}S_{xx} + \frac{1}{Y}S_{yy} - \frac{\sigma}{Y}S_{zz}$$

BUT FOR THIS SYSTEM:

$$S_{yy} = S_{zz} = 0$$

HENCE:

$$\epsilon_{xx} = \frac{1}{Y}S_{xx}$$
$$\epsilon_{yy} = -\frac{\sigma}{Y}S_{xx} = -\sigma\epsilon_{xx}$$
$$= -\frac{\sigma}{Y}\rho g(\ell_0 - x)$$

b) $d\xi = \epsilon_{xx}dx$

$$= \frac{1}{Y}\rho g(\ell_0 - x)$$

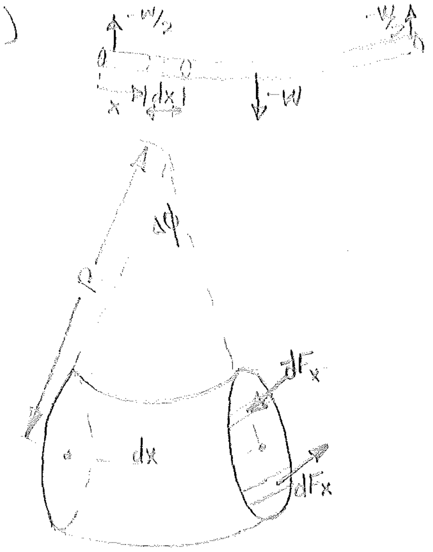
INTEGRATING OVER X FROM 0 TO ℓ_0 GIVES THE AMOUNT THE HANGING ROD HAS STRETCHED AT ℓ_0.

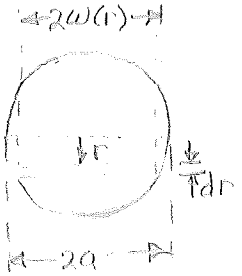
$$\xi(\ell_0) = \frac{1}{Y}\rho g\int_0^{\ell_0}(\ell_0 - x)dx$$
$$= \frac{\rho g}{Y}\left(\ell_0 x - \frac{x^2}{2}\right)\Big|_{x=\ell_0}$$
$$= \frac{\rho g}{2Y}\ell_0^2$$

THE RODS ENTIRE LENGTH WHEN STRETCHED IS:

$$\ell = \ell_0 + \xi(\ell_0)$$
$$= \ell_0\left[1 + \frac{\rho g}{2Y}\ell_0\right] \quad ✓$$

(1-8)





$$\omega^2 + r^2 = a^2$$
$$\Rightarrow \omega(r) = (a^2 - r^2)^{1/2}$$

AS WITH THE RECTANGULAR CROSS SECTION: $\epsilon_{xx} = r/R$

THUS: $\frac{dF_x}{2\omega dr} = \frac{Yr}{R} \Rightarrow dF_x = \frac{2Y}{R} \omega r dr$

$$= \frac{2Y}{R}(a^2 - r^2)^{1/2} r dr$$

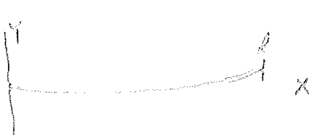
THE TORQUE FROM dF_x AND $-dF_x$ IS:

$$d\tau_z = 2r\, dF_x$$
$$= \frac{4Y}{R}(a^2 - r^2)^{1/2} r^2 dr$$

SUMMING THESE TORQUES ON FACE YIELDS BENDING MOMENT M:

$$M = \int d\tau = \frac{4Y}{R}\int_0^a r^2 (a^2 - r^2)^{1/2} dr$$
$$= \frac{4Y}{R}\left[-\frac{r}{4}(a^2 - r^2)^{3/2} + \frac{a^2}{8}\left\{ r(a^2 - r^2)^{1/2} + a^2 \sin^{-1}\frac{r}{|a|} \right\} \right]_0^a$$
$$= \frac{4Y}{R}\left[\left(\frac{a^2}{8}\right)\left(\frac{a^2 \pi}{2}\right) \right]$$
$$= \frac{Y \pi a^4}{4R}$$

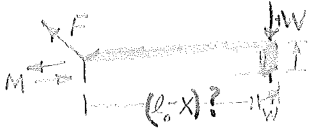
NOW $R = \left[\frac{d^2 Y}{dX^2}\right]^{-1}\left[1 + \left(\frac{dY}{dX}\right)^2\right]^{3/2}$, BUT Y DISPLACEMENT IS SO SMALL, $\frac{dY}{dX} \simeq 0 \Rightarrow R \simeq \left[\frac{d^2 Y}{dX^2}\right]^{-1}$



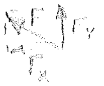
$$\therefore M = \frac{Y \pi a^4}{4}\frac{d^2 Y}{dX^2}$$ ✓

(1-9)

(a)

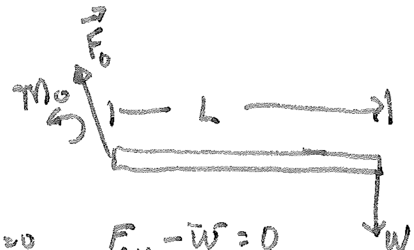


$M = -(\ell_0 - X)W \quad (\ell \times F)_3 =$



$F_X = 0$

$F_Y = -W$ ✓



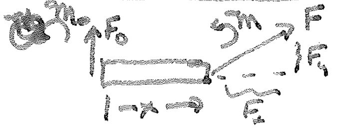
$\Sigma F_y = 0 \quad F_{0y} - \bar{W} = 0$

$\Sigma F_x = 0 \quad F_{0x} = 0$

$\Sigma \tau_z = 0 \quad \mathcal{M}_0 - \bar{W}L = 0$

$\Rightarrow F_{0y} = \bar{W}$

$\mathcal{M}_0 = \bar{W}L$



$\Sigma F_x = 0 \quad F_x + F_{0x} = 0 \Rightarrow F_x = 0$

$\Sigma F_y = 0 \quad F_y + F_{0y} = 0 \Rightarrow F_y = -W$

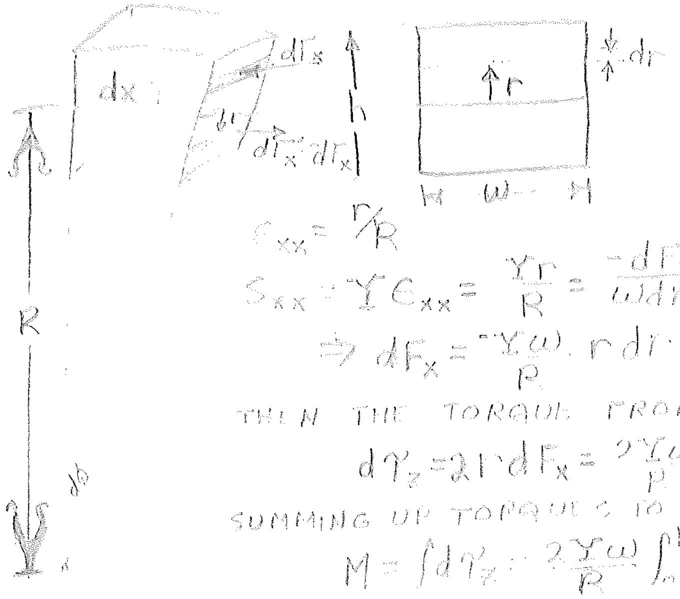
$\Sigma \tau_z = 0 \quad \mathcal{M}_0 + F_y x + \mathcal{M} = 0$

$\mathcal{M} = -\mathcal{M}_0 + Wx$

$= -\bar{W}L + Wx$

$= W(x - L)$

(b) ISOLATING SMALL SEGMENT OF THE BEAM:



$\epsilon_{xx} = r/R$

$S_{xx} = Y\epsilon_{xx} = \frac{Yr}{R} = \frac{-dF_x}{w\,dr}$

$\Rightarrow dF_x = -\frac{Yw}{R}\, r\,dr$

THEN THE TORQUE FROM dF_x & dF'_x IS:

$d\tau_z = 2r\,dF_x = \frac{2Yw}{P} r^2\,dr$

SUMMING UP TORQUES TO FIND BENDING MOMENT:

$M = \int d\tau_z = \frac{2Yw}{R}\int_0^{h/2} r^2\,dr$

$= \frac{2Yw}{R}\,\frac{(h/2)^3}{3}$

correct but I don't follow how you got it

$M = \frac{Ywh^3}{12R}$ (THE SAME AS TEXT'S EXAMPLE)

NOW $M = -(\ell_0 - X)W = \frac{Ywh^3}{12R} \cong \frac{Ywh^3}{12}\frac{d^2Y}{dX^2}$ FOR SMALL dY/dx

$\Rightarrow \frac{d^2Y}{dx^2} = \frac{-12W}{Ywh^3}(\ell_0 - x)$

$\frac{dY}{dx} = \frac{-12W}{Ywh^3}\left(\ell_0 X - \frac{X^2}{2} + C_1\right)$

@ $X = 0,\ \frac{dY}{dx} = 0 \Rightarrow C_1 = 0$

$Y = \frac{-12W}{Ywh^3}\left(\frac{\ell_0 X^2}{2} - \frac{X^3}{6} + C_2\right)$

@ $X = 0;\ Y = 0 \Rightarrow C_2 = 0$

$\therefore\ Y = \frac{-2W}{Ywh^3}\left(3\ell_0 X^2 - \text{②} X^3\right)$

CHAPTER II PROBLEMS

2.1 The solution of (2.2) can be written in the two equivalent forms:

$$x = C\cos\omega_0 t + D\operatorname{sen}\omega_0 t$$

or

$$x = A\cos(\omega_0 t + \phi)$$

Find A and ϕ in terms of C and D.

2.2 A particle executing simple harmonic motion is observed to have a speed of 3 cm/sec at the instant it passes the midpoint of its path. If the frequency f_o of the oscillation is 10 hertz write an expression of the form (2.4) which will correctly describe the motion of this particle. Assume the particle is moving along the x-axis with the origin at the midpoint of the path, and that one starts counting time at the instant the particle is passing the midpoint and moving to the right.

✓ 2.3 The real part of

$$\underset{\sim}{x}(t) = 4e^{i\pi t}$$

is a description of a particle executing simple harmonic motion. (a) What is the real part of this expression? (b) What is the frequency of the oscillation? (c) What is amplitude? (d) Plot x(t) in the complex plane at times t = 0, t = 1/4, t = 1/2, t = 1 sec. What is the angular velocity of the point (or vector) representing x(t)?

2.4 The real parts of

$$\underset{\sim}{x}(t) = 4e^{\underset{\sim}{i}\pi t}$$

and

$$\underset{\sim}{x}_1(t) = (3+4\underset{\sim}{i})\, e^{\underset{\sim}{i}\pi t}$$

represent simple harmonic motions. Do they have the same frequency? The same amplitude? Represent $\underset{\sim}{x}(t)$ and $\underset{\sim}{x}_1(t)$ in the complex plane at t = 0. What is the phase difference between $\underset{\sim}{x}(t)$ and $\underset{\sim}{x}_1(t)$? Which leads?

2.5 If $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$ represent two simple harmonic motions of the same frequency and if

$$\underset{\sim}{x}_1 = \frac{\underset{\sim}{x}_2}{1+\underset{\sim}{i}}$$

find the phase difference between $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$. Which leads? Find the ratio of the amplitude of $x_1(t)$ to that of $x_2(t)$.

2.6 If $x_1(t)$ and $x_2(t)$ represent two simple harmonic motions of the same frequency and if

$$\underset{\sim}{x}_1(2+\underset{\sim}{i}) = \underset{\sim}{x}_2(1-\underset{\sim}{i})$$

Chapter I - ELEMENTS OF ELASTICITY

The study of acoustics is basically a study of vibrations and waves. Practically all solids and fluids are elastic in the sense that the application of external forces to a small portion of a solid or fluid produces a distortion of that portion and gives rise to internal forces which tend to restore that portion to its original undistorted state. If the external forces are removed suddenly, an oscillation of the small portion generally ensues. This is transmitted to the neighboring portion of the medium, which in turn transmits it to their neighboring portions. We speak of this process as wave propagation. The nature of the waves and the speed with which they are propagated are intimately related to what are referred to as the elastic properties of the medium. Consequently, it will be appropriate to begin our study of acoustics by reviewing the basic concepts of elasticity.

1.1 Stress and Strain

If a long wire is suspended vertically from a fixed support and its length and diameter are measured for a number of different kilogram masses hung from its lower end (Fig. 1.1a), one finds that the length increases and the diameter decreases linearly with the force mg exerted on the wire, as indicated in Fig. 1.16.*
If the experiment is repeated with a number of wires of different lengths and diameters, but all made from the same material, then

---

* The linear relation between the length or diameter of the wire and the force exerted on it is observed only over a limited range of forces ranging from zero to some maximum value which depends on the diameter of the wire and the material from which the wire is made. In all that follows it is assumed that the force always lies within this range.

then for *each* wire one obtains the linear relationship shown in Fig. 1.1b.. The slopes and intercepts, however, are in general different for each wire. If, instead of plotting the length l and diameter d as a function of the applied force, one plots $(l-l_0)/l_0$, and $(d-d_0)/d_0$ against F/A where A is the cross-sectional area of the wire, one obtains *identical graphs for all wires made of the same material*. (Fig. 1.2). The quantities $(l-l_0)/l_0$, $(d-d_0)/d_0$, and F/A thus appear to be more useful quantities than l, d and F in describing the behavior of the material. The ratios $(l-l_0)/l_0$ and $(d-d_0)/d_0$ are called *strains*, while the ratio F/A is called a *stress*.

The relation between the stress and the corresponding strain depicted in Fig. 1.2 can be represented by the equations

$$\left.\begin{aligned} \frac{l-l_0}{l_0} &= \frac{1}{Y}\frac{F}{A} \\ \frac{d-d_0}{d_0} &= -\frac{\sigma}{Y}\frac{F}{A} \end{aligned}\right\} \qquad (1.1)$$

where Y and σ are constants. These constants are *characteristic of the material* from which the wire is made, and are called *Young's modulus* and *Poisson's ratio* respectively. Typical values of these constants for a few materials are shown in Table 1.1.

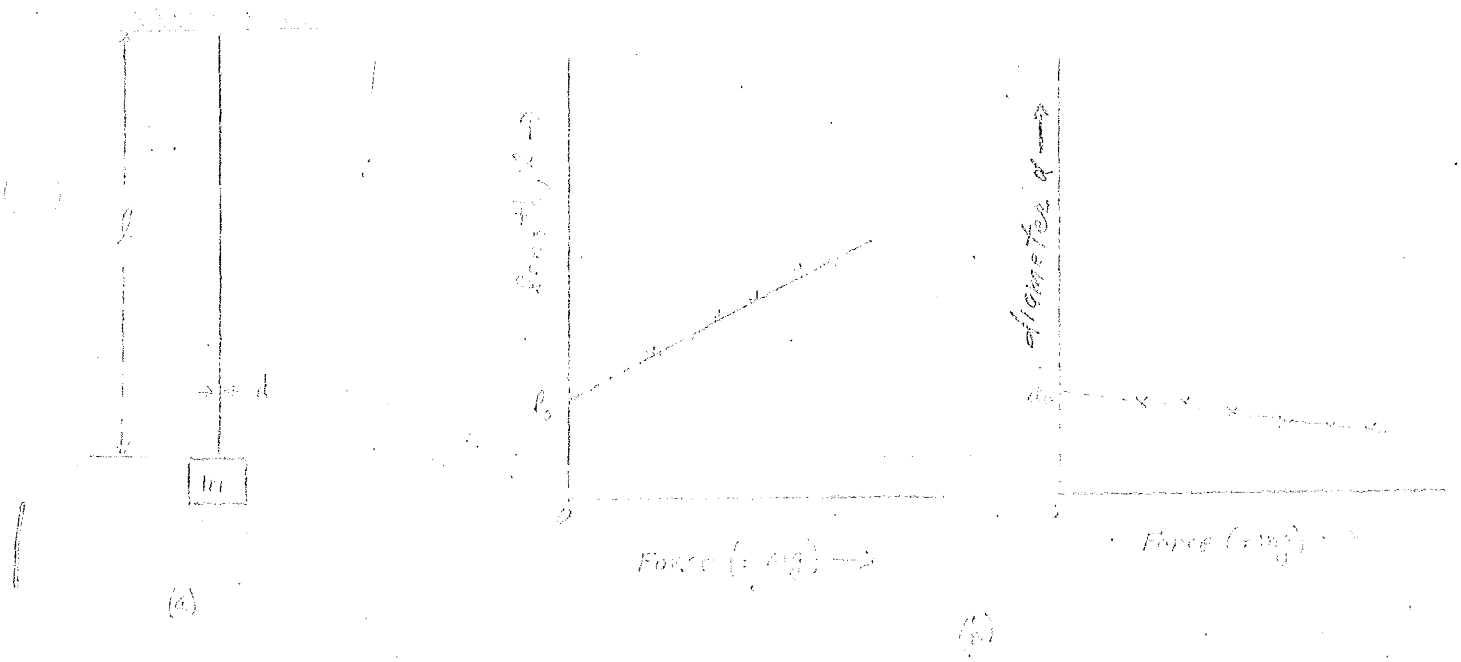

ℓ
d
m
(a)
length ℓ →
ℓ₀
0
Force (= mg) →
diameter d →
d₀
0
Force (= mg) →
(b)

Fig 1.1

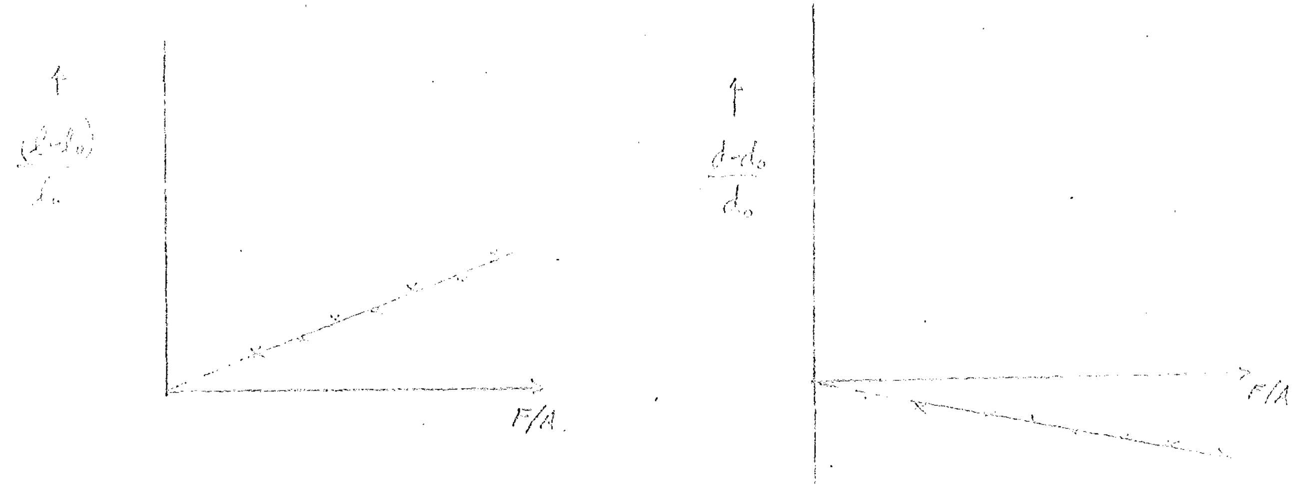

↑
(d-d₀)/d₀
F/A
↑
(d-d₀)/d₀
F/A

Fig 1.2

TABLE 1.1

Substance	Young's Mod. Y nts/m	Poisson's Ratio,	Bulk Modulus nts/m²	Shear Modulus nt/m²
Aluminum	7×10^{10}	0.35	7×10^{10}	$2{,}7 \times 10^{10}$
Beryllium	31×10^{10}	0.05		
Brass	10×10^{10}	0.37	13×10^{10}	4×10^{10}
Copper	12×10^{10}	0.37	15×10^{10}	5×10^{10}
Iron	20×10^{10}	0.29	16×10^{10}	8×10^{10}
Pyrex Glass	6×10^{10}	0.24	4×10^{10}	$2{,}5 \times 10^{10}$
Lucite	0.4×10^{10}	0.4	$0{,}7 \times 10^{10}$	$0{,}1 \times 10^{10}$

The two constants, Y and σ, are sufficient to completely describe the elastic behavior of homogeneous isotropic materials.* The large numerical value of Y ($\sim 10^{11}$ nts/m^2) suggests that in the majority of cases encountered, the strains are very small quantities. For example a 10 KG mass hung on the end of a 1 mm diameter brass wire will result in a strain, $(l - l_0)/l_0 = 1.3 \times 10^{-3}$. In what follows, we assume the strains are small compared to unity.

Other experiments indicate that equations (1.1) are somewhat more general. If a rectangular block of dimensions l_0, w_0, and h_0 is subjected to equal and opposite forces applied to any two opposite faces, the changes which occur in any of the dimensions can be expressed by equations of the form (1.1). For example, if F stands for the magnitude of the resultant of the set of forces acting on either end face of the block shown in Fig. 1.3a, and A the area of one of the end faces then the experimental results indicate that

$$\left.\begin{aligned} &\frac{l-l_0}{l_0} = \frac{1}{Y}\frac{F}{A} \\ \text{and} \\ &\frac{w-w_0}{w_0} = \frac{h-h_0}{h_0} = -\frac{\sigma}{Y}\frac{F}{A} \end{aligned}\right\} \quad (1.2)$$

Here l, w, h refer to the length, width and height of the block, after the forces are applied and l_0, w_0, h_0, to those same quantities before the forces are applied.

---

* A homogeneous substance is one whose physical properties are the same at all points of the body. An isotropic substance is one whose physical properties at a point are independent of direction.

If forces are applied to the top and bottom faces as in Fig. 1.3(b), then the results indicate that

$$\left.\begin{aligned} \frac{h-h_0}{h_0} &= \frac{1}{Y}\frac{F'}{A'} \\ \frac{w-w_0}{w_0} &= \frac{l-l_0}{l_0} = -\frac{\sigma}{Y}\frac{F'}{A'} \end{aligned}\right\} \qquad (1.3)$$

where F' is the resultant of the set of forces acting on one of the faces of area A'. If the *direction* of the two sets of forces in either Fig. 1.3a or b is reversed, the *signs* of the righthand terms of equations (1.2) or (1.3) is changed. If the set of forces shown in Fig. 1.3a and the set shown in Fig. 1.3b are applied *simultaneously*, it is found that the principle of superposition* holds, i.e.

$$\frac{l-l_0}{l_0} = \frac{1}{Y}\frac{F}{A} - \frac{\sigma}{Y}\frac{F'}{A'}$$

$$\frac{w-w_0}{w_0} = -\frac{\sigma}{Y}\frac{F}{A} - \frac{\sigma}{Y}\frac{F'}{A'}$$

$$\frac{h-h_0}{h_0} = -\frac{\sigma}{Y}\frac{F}{A} + \frac{1}{Y}\frac{F'}{A'}$$

---

* The strain produced by n sets of forces acting simultaneously is the resultant of the strains produced by each set acting separately.

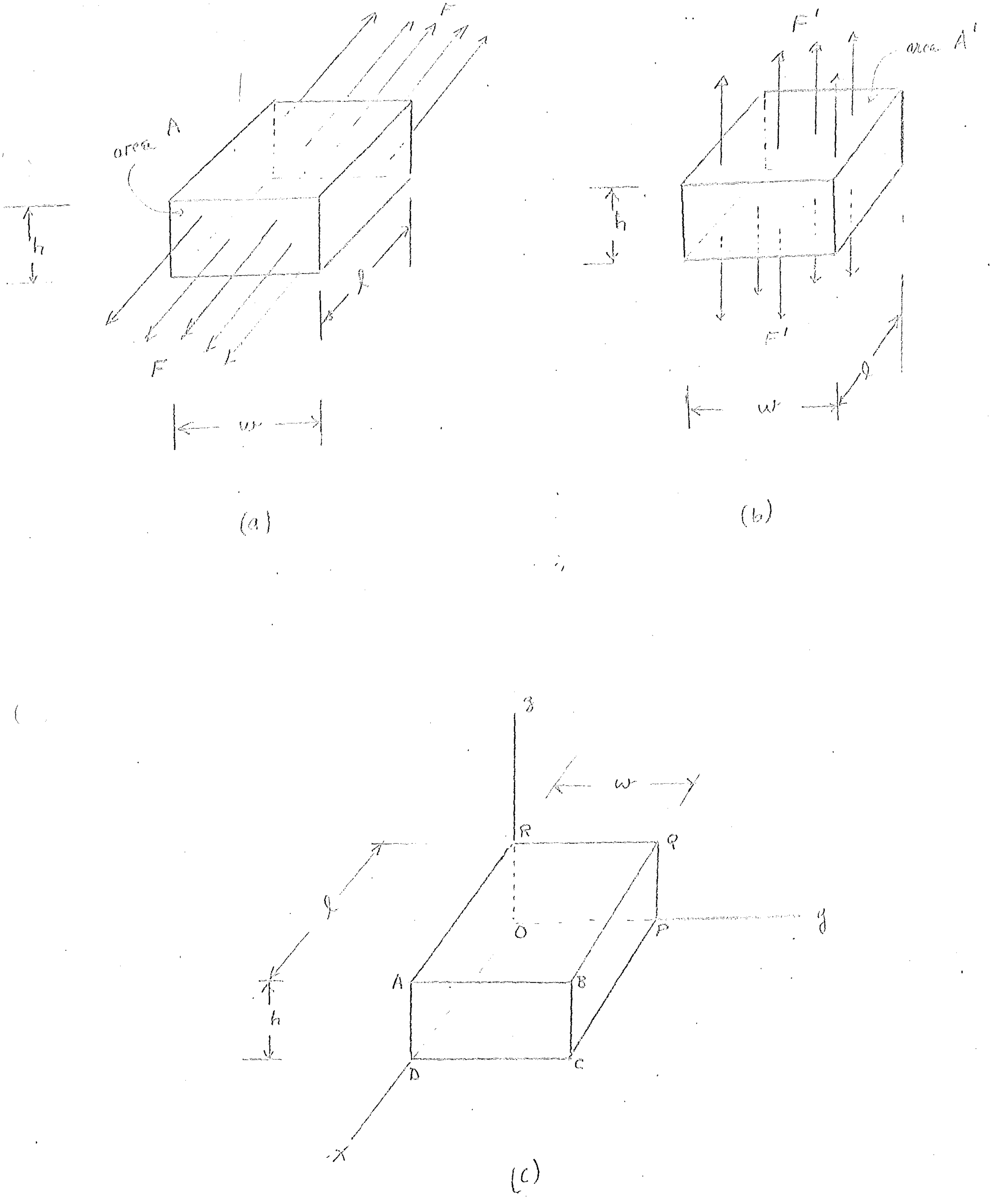


Fig 1.3

By using a coordinate system such as that shown in Fig. 1.3c, the results of all experiments of this general nature can be summarized conveniently by the equations

$$\left.\begin{aligned} \epsilon_{xx} &= \frac{1}{Y} S_{xx} - \frac{\sigma}{Y} S_{yy} - \frac{\sigma}{Y} S_{zz} \\ \epsilon_{yy} &= -\frac{\sigma}{Y} S_{xx} + \frac{1}{Y} S_{yy} - \frac{\sigma}{Y} S_{zz} \\ \epsilon_{zz} &= -\frac{\sigma}{Y} S_{xx} - \frac{\sigma}{Y} S_{yy} + \frac{1}{Y} S_{zz} \end{aligned}\right\} \qquad (1.4)$$

Here

$$S_{xx} = \frac{\text{x-component of the resultant force acting on face ABCD}}{\text{area of face ABCD}}$$

$$S_{yy} = \frac{\text{y-component of the resultant force acting on face BCPQ}}{\text{area of face BCPQ}}$$

$$S_{zz} = \frac{\text{z-component of the resultant force acting on face ABQR}}{\text{area of face ABQR}}$$

As before ϵ_{xx}, ϵ_{yy}, ϵ_{zz} are called strains, S_{xx}, S_{yy}, and S_{zz} are called stresses. Although it is assumed that equal and opposite forces are applied to a given pair of opposite faces, note that the stresses are defined in terms of the forces acting on faces ABCD, BCPQ, and ABQR. These are the "positive" faces of the block in the sense that an outwardly drawn normal to any one of these faces points in the positive direction of one of the coordinate axes. It should be apparent that the stresses and the strains are algebraic quantities. S_{xx}, for example, is positive if the forces acting on face ABCD are directed out of the block, and negative if the forces are directed into the block.

In the examples given above it was assumed that the external forces were <u>zero</u> initially and that the strains resulted from the application of external forces producing the stresses S_{xx}, S_{yy}, S_{zz}. In many cases of interest, one is interested in the strains that occur when the external forces are <u>changed</u> from one set to another. For example, suppose as in Fig. 1.4 a rod has a length ℓ_1 when subjected to equal and opposite forces of magnitude F_1 and a length ℓ_2 when subjected to forces of magnitude F_2. If the unstressed length is ℓ_0, one can write using equations (1.4)

$$\ell_1 = \ell_0 \left[1 + F_1/AY\right]$$

$$\ell_2 = \ell_0 \left[1 + F_2/AY\right]$$

where A is the cross-section of the rod. Subtracting and rearranging one obtains

$$\frac{\ell_2 - \ell_1}{\ell_0} = \frac{F_2 - F_1}{AY} \cong \frac{\ell_2 - \ell_1}{\ell_1}.$$

since the difference between ℓ_0 and ℓ_1 is very small. One interprets $(\ell_2 - \ell_1)/\ell_1$ as the strain resulting from the change $\Delta F = F_2 - F_1$ in the external forces. In like fashion, ϵ_{xx}, ϵ_{yy}, and ϵ_{zz} in equations (1.4) can be interpreted as the strains resulting from <u>changes</u> in the stresses of amounts S_{xx}, S_{yy}, S_{zz}.

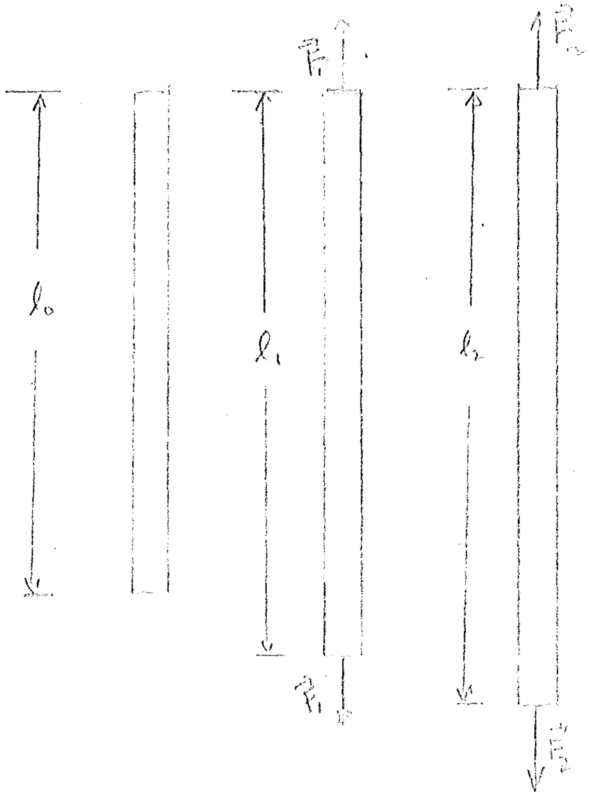


Fig 1.4

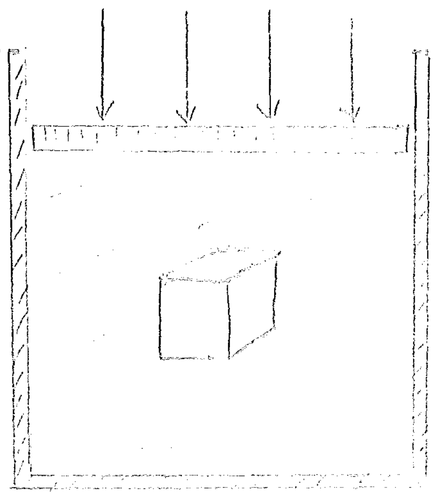


Fig 1.5

2.1 Bulk Modulus

If a block is subjected to a uniform pressure by placing it for example in a pressure tank containing some ~~fluid~~ liquid as in Fig 1.5, it is found experimentally that any change ΔP in the pressure results in a corresponding change, ΔV, in the volume of the block such that the ratio of the change in pressure to the change in volume per unit volume is a constant. This constant

$$B = -\frac{\Delta P}{\Delta V/V} \qquad (1.5)$$

is called the bulk modulus of the material from which the block is made. If the experiment is carried out in such a manner that the block is maintained at constant temperature during the experiment, the constant ratio is called the isothermal bulk modulus. If the changes in pressure and the corresponding measurements of the changes in volume are made sufficiently rapidly so that during this time there is negligible heat transfer between the block and the fluid, a different constant called the adiabatic bulk modulus is obtained.

It was stated earlier that the two constants Y and σ are sufficient to describe the elastic behavior of homogeneous isotropic materials. The bulk modulus, B, must therefore be related to Y and σ. One can derive this relationship by applying equations (1.4) to a block and subjected to a ~~uniform pressure P~~ hydrostatic pressure as in Fig 1.5. For convenience let V_0 be the volume of the block when the ~~block is in a vacuum and the pressure is zero,~~ pressure is P and let V' be the volume when the block is subjected to a pressure P'. Remembering that pressure is a force per unit area, and that the forces on a surface due to pressure are always in the nature of ~~presses~~ Pushes, it should be apparent that when the pressure is P

$$S_{xx} = S_{yy} = S_{zz} = -P$$

and when the pressure is P'

$$S_{xx} = S_{yy} = S_{zz} = - P'$$

Interpreting ϵ_{xx}, ϵ_{yy} and ϵ_{zz} of equations (1.4) as the strains due to the change in pressure from P to P' one obtains

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = \frac{1}{Y} (2\sigma - 1) (P - P')$$

Letting ℓ, w and h stand for the dimensions of the block when the pressure is P, ℓ', w', h', the dimensions of the block when the pressure is P' one has from the definitions of ϵ_{xx}, ϵ_{yy}, and ϵ_{zz}

$$\begin{aligned} V' - V &= \ell' w' h' - \ell wh \\ &= \ell(1+\epsilon_{xx}) w(1+\epsilon_{yy}) h(1+\epsilon_{zz}) - \ell wh \\ &= V \left[(1+\epsilon_{xx})^3 - 1 \right]. \end{aligned}$$

Since in almost all cases $\epsilon_{xx} \ll 1$ we have as a good approximation

$$V' \doteq V = V \left[(1+3\epsilon_{xx}) - 1 \right]$$

so that

$$(V'-V)/V = 3\epsilon_{xx} = \frac{3}{Y} (2\sigma - 1) (P-P')$$

and

$$B = - \frac{P'-P}{(V'-V)/V} = \frac{Y}{3(1-2\sigma)} \qquad (1.6)$$

For all materials, B and Y are positive. Equation (1.6) suggests therefore that σ must be less than 1/2, a result that is confirmed experimentally.

1.3. Shearing stresses and strains, shear modulus

Consider a block subjected to the set of forces illustrated in Fig. 1.6a. As in our earlier examples, the forces acting on any one face are equal and opposite to the forces acting on the opposite face (this is necessary for the block to be in translational equilibrium). Forces which are tangential to a surface such as those shown in the figure are referred to as shearing forces and the quantities

$$S_{yz} = \frac{\text{z-component of the resultant force acting on face BCGF}}{\text{area of face BFGD}}$$

and

$$S_{zy} = \frac{\text{y-component of the resultant force acting on face ABFE}}{\text{area of face ABFE}}$$

are referred to as shearing stresses.* For the block to be in rotational equilibrium (consider, for example, torques about the x-axes) S_{yz} must equal S_{zy}. Under the action of the set of shearing forces shown in Fig. 1.6a, the block is deformed into a parallelepiped as indicated by the solid lines in Fig. 1.6b. The angle Θ (in radians) is referred to as the shearing strain, and the ratio of the shearing stress to the shearing strain is called the shear modulus G, i.e.

$$G = \frac{S_{yz}}{\Theta} \qquad (1.7)$$

For many materials, this ratio is found to be constant over a reasonably wide range of stresses. Because of the large numerical value of G (see table 1.1), the strain Θ is usually small compared to unity.

---

* The reason for the double subscript on the stresses should now be clear. The first subscript identifies the face on which the force is acting, while the second specifies which component of the force is involved. For example, S_{xy} refers to the y-component of the force acting on the face which is perpendicular to the x-axis.

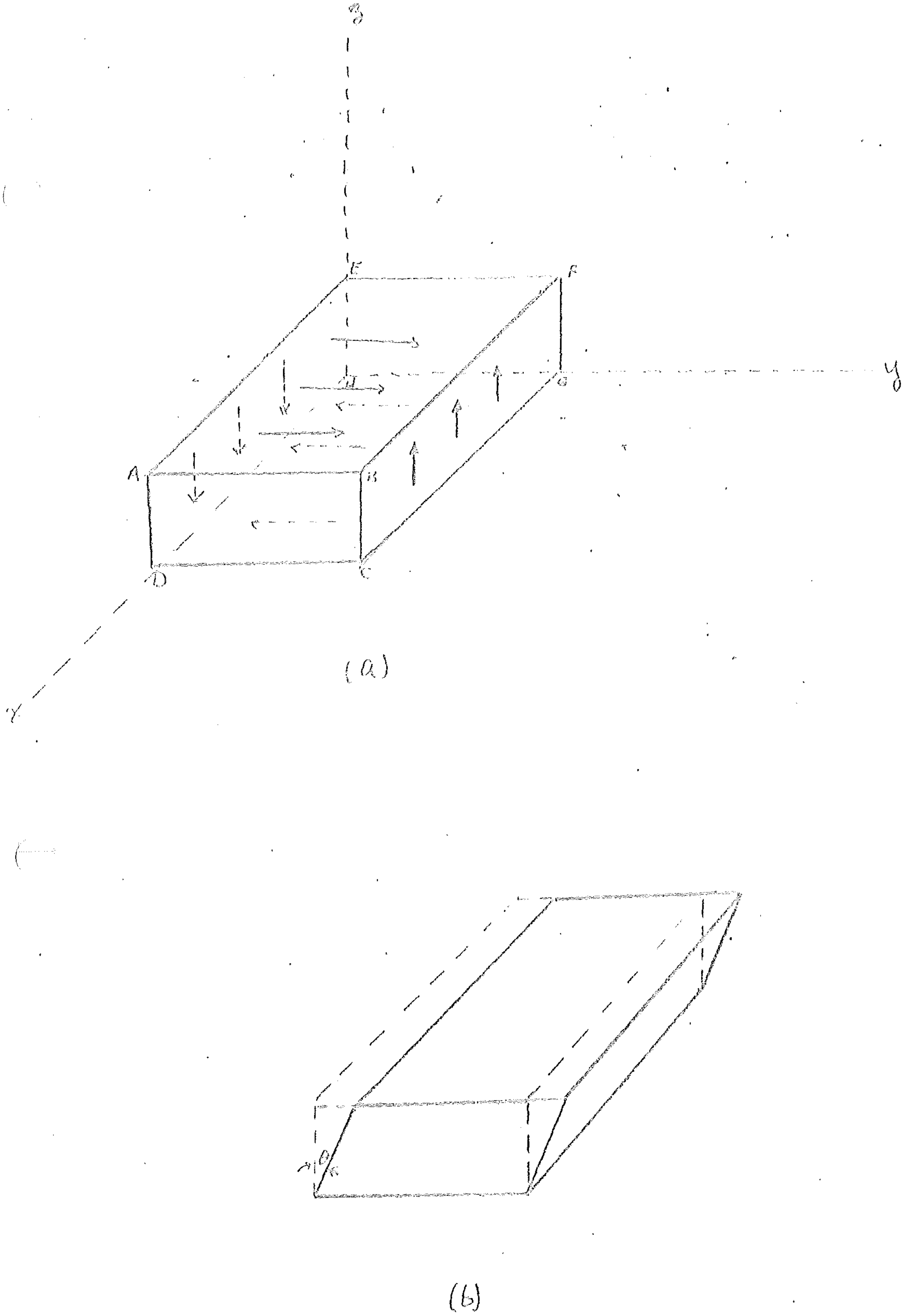


Fig 1.6

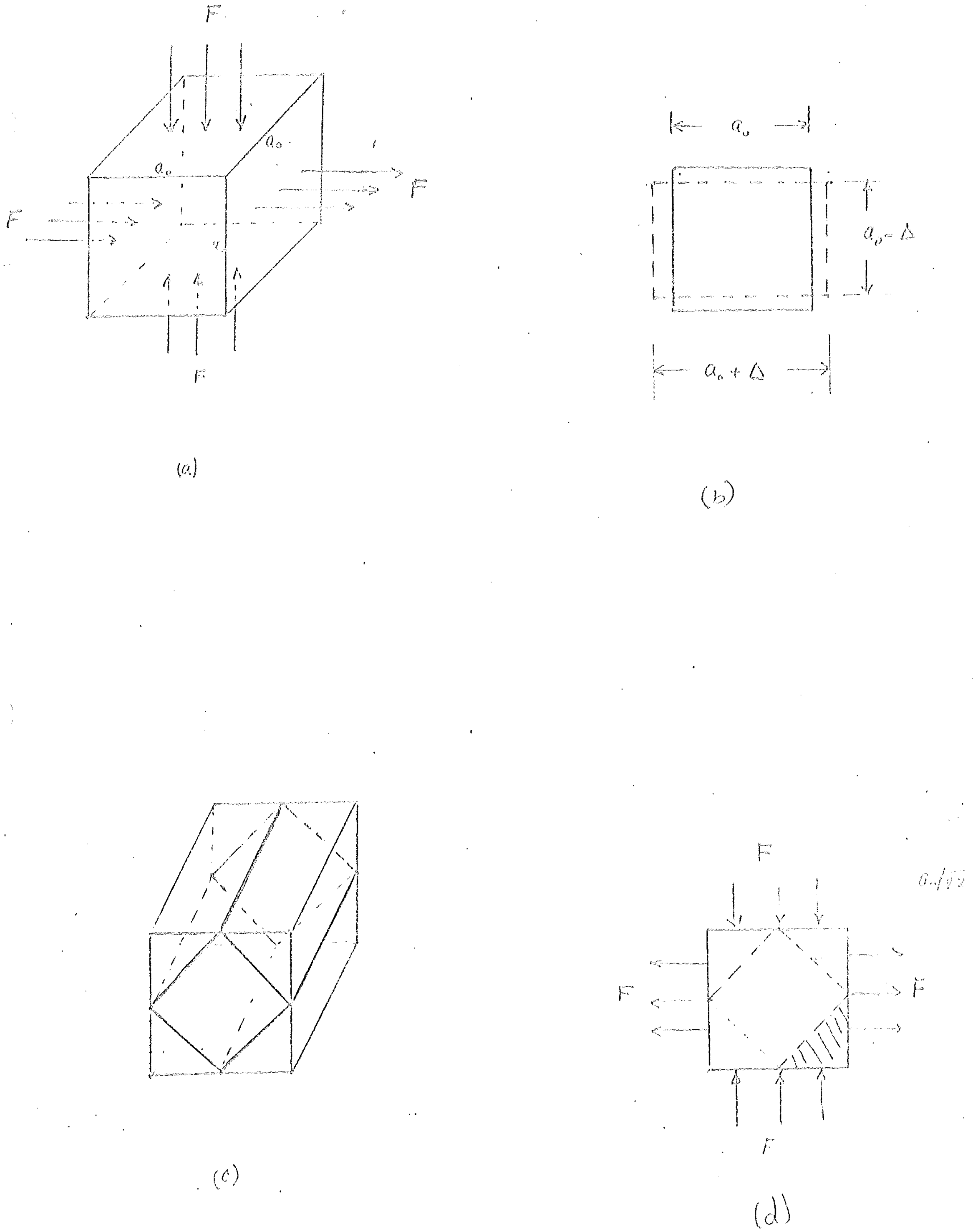
It is not very difficult to show that the shear modulus can be expressed in terms of Y and σ . Consider a block in the form of a cube of edge a_0 and subject it to the set of forces shown in Fig. 1.7a. Let the resultant of the forces acting on each of the four faces be F and let $A = a_0^2$ be the area of one of the faces. Using equations (1.4) one finds that the height is shortened and the width is increased by an amount

$$\Delta = \frac{F a_0}{AY}\left[1+\sigma\right]$$

as indicated in Fig. 1.7b which shows only the front face of the cube. After the distortions occur all portions of the block are in equilibrium and if one isolates <u>any portion</u> of the block it will be in equilibrium under the action of forces exerted by the material adjacent to the isolated portion. We inquire into the nature of the forces exerted <u>on</u> that portion of the block bounded by the rectangular parallelepiped shown in red in Fig. 1.7c. The front face of the rectangular parallelepiped is shown by the dotted lines in Fig. 1.7d. Isolating the triangular portion of the cube shown by the shaded area and drawing in the forces* acting on it (Fig. 1.8a), it should be evident that for this triangular portion to be in equilibrium, the resultant, F_s, of the forces acting on the slant face must be <u>tangential</u> to the surface as indicated and must be equal in magnitude to $F/\sqrt{2}$.

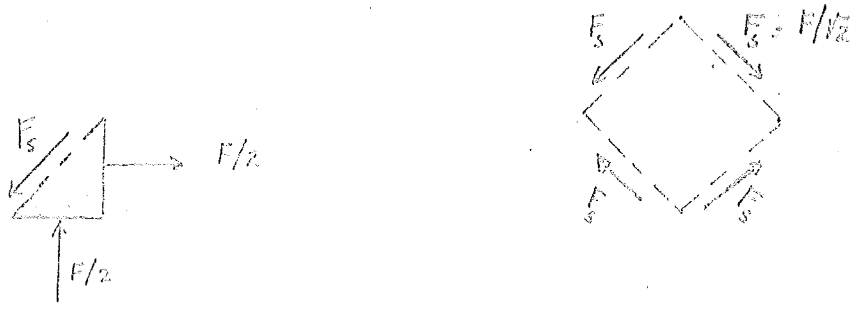
---

* When using equilibrium conditions to calculate the <u>internal forces</u> (and stresses) that arise when a block is subjected to external forces, one often ignores the distortions that are produced and calculates the internal forces <u>as if there were no distortions.</u> This procedure yields satisfactory results as long as the distortions (strains) are small compared to unity.



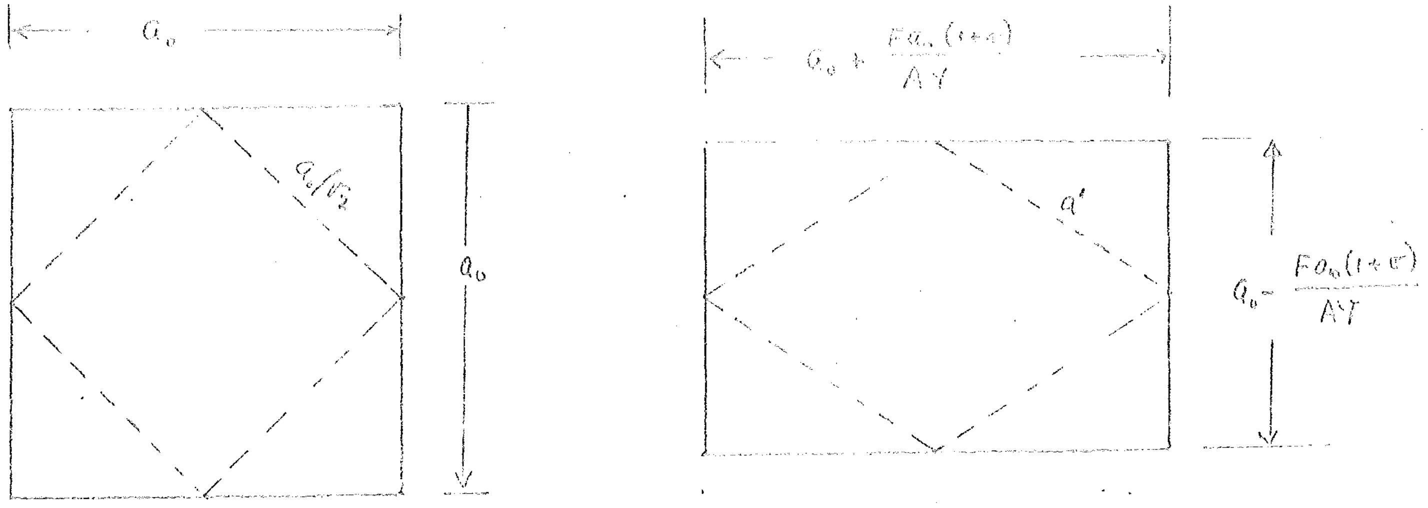
(a) (b) (c) (d)

Fig 1.7



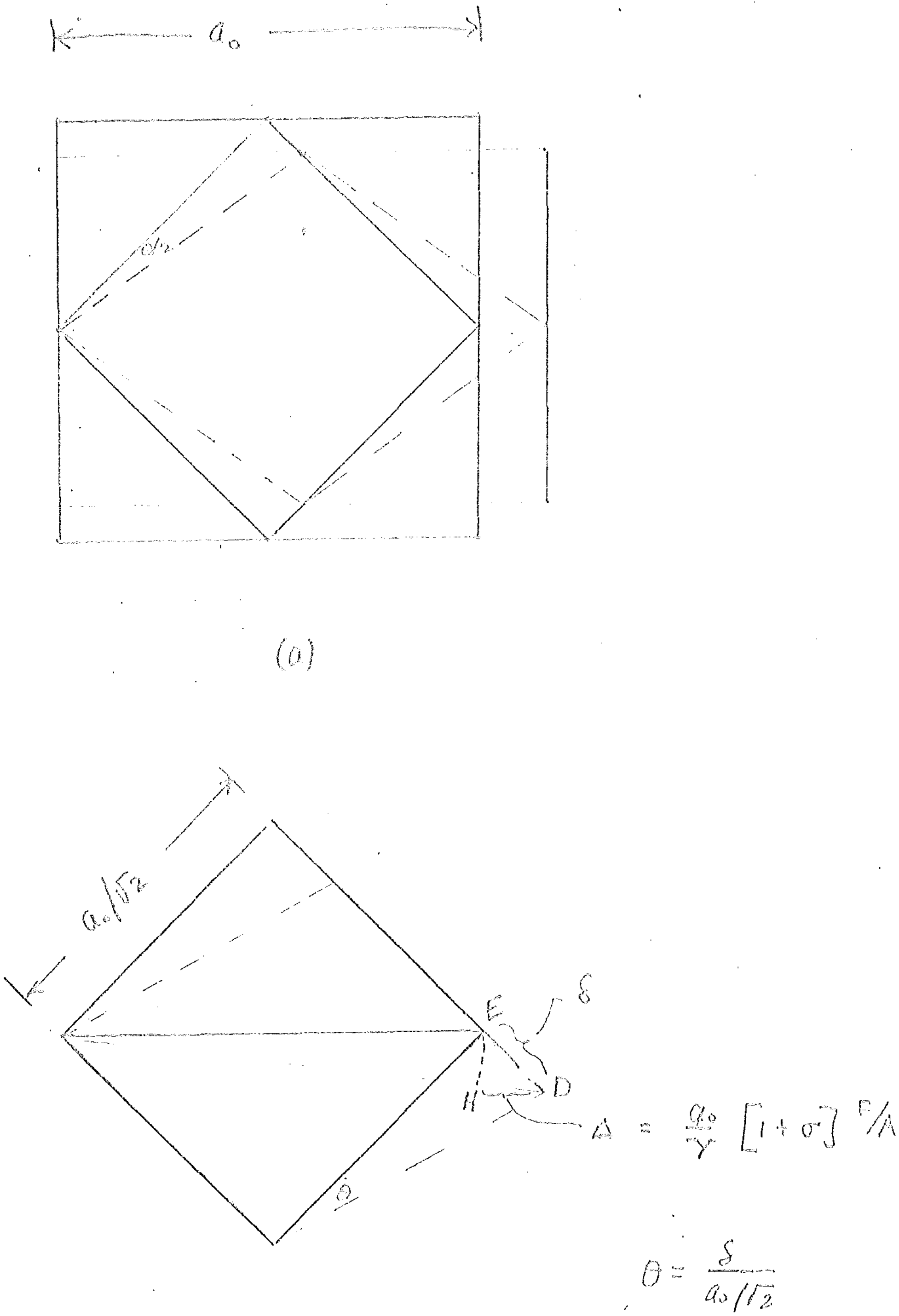
(a) (b)

Fig 1.8



(a) (b)

Fig 1.9


a_0
α/2
$a_0/\sqrt{2}$
E
δ
D
H
θ
$\Delta = \frac{a_0}{\gamma}[1+\sigma]\,F/A$
$\theta = \frac{\delta}{a_0/\sqrt{2}}$

(a)

(b)

Fig 1.10

Similarly, by isolating the other three triangular sections and using Newton's third law one can conclude that the forces exerted on the rectangular parallelepiped are the forces shown in Fig. 1.8b. The area, A_s, of one of the side faces of the parallelepiped is equal to $\frac{a_0^2}{\sqrt{2}}$ or $A/\sqrt{2}$, and since $F_s = F/\sqrt{2}$ it follows that the shearing stress F_s/A_s at the side face is numerically equal to the (normal) stress F/A at the surface of the cube. Note that the arrangement of the shearing forces on the faces of the parallelepiped is exactly the same as the shearing forces shown acting on the block of Fig. 1.6a; consequently, these shearing forces should produce some shearing strain, θ, which in this instance can be calculated in terms of Y and σ.

Figures 1.9a and b illustrate the distortions produced in the rectangular parallelepiped when the forces are applied to the cube. The end faces of the parallelepiped which were originally square become parallelograms. In Fig. 1.10a, the original square face (red lines) and the distorted end face (dashed lines) are shown with the left edge superimposed and Fig. 1.10b shows these two faces after the original square face has been rotated through an angle of θ/2 with respect to the dashed face. From Fig. 1.9b the increase, Δ, in the length of the diagonal of the distorted face $Fa_0(1+\sigma)/AY$. Since $\theta \ll 1$, the angle HDE in Fig. 1.10b is very nearly equal to 45°. Hence from the figure

$$\delta = \frac{\Delta}{\cos 45^\circ} = \frac{Fa_0(1+\sigma)}{AY}\sqrt{2}$$

and

$$\theta = \frac{\delta}{a_0/\sqrt{2}} = \frac{F(1+\sigma)2}{AY}$$

and

$$\frac{F/A}{\theta} = \frac{F_s/A_s}{\theta} = G = \frac{Y}{2(1+\sigma)} \qquad (1.8)$$

This equation expresses the relationship between the shear modulus, G, and Young modulus, Y, and Poisson's ratio σ .

4. Stress and strain at a point

In section 3 we have seen how external forces acting on a cubical block give rise to stresses on the surfaces inside the block. The stress at any point of the block can be defined in terms of the stresses on the faces of an infinitesimal surface containing the point.* Similarly one can define the strain at a point of the block in terms of the distortions taking place in a small volume surrounding the point. To illustrate how one determines the stress and strain at a point we consider a thin rod which is hung from one end as in Fig. 1.11a. Let the rod be uniform of density P and mass m and have a length l_0, width w_0, and thickness h_0 when unstressed (e.g. when resting on a horizontal table). When the rod is hung from one end, its length will increase slightly due to the stresses set up by the gravitational force. We wish to determine the stress at some general point P located a distance x from the supported end. First it should be evident that since the entire rod is in equilibrium, the force

---

* If one chooses the surface to be a rectangular parallelepiped whose edges are parallel with the axes of a rectangular coordinate system, then the resultant force acting on each "positive" face of the surface can be resolved into three components. Since there are three positive faces, there are nine stress components, S_{xx}, S_{xy}, S_{xz}, S_{yx}, S_{yy}, S_{yz}, S_{zx}, S_{zy}, S_{zz}. These nine components form what is called the stress tensor. The strain at a point is similarly described by nine strain components, forming what is called the strain tensor.

exerted by the support must equal mg, the weight of the rod. If one isolates the portion of the rod between the support and point P, as indicated in Fig. 1.11b, the forces acting on this portion are the force exerted by the support, the gravitational force, and a force ~~labeled~~ labeled $\vec{F}$, which represents the force exerted by the lower portion of the rod. Since the isolated portion of the rod is in equilibrium we must have

$$F_x = mg - \rho\,[h_o w_o x]\,g$$

where F_x is the x-component of $\vec{F}$. If we let the cross-section at P be the bottom surface of a small rectangular parallelepiped containing P (Fig. 1.11c) this bottom surface is a positive face of the parallelepiped and

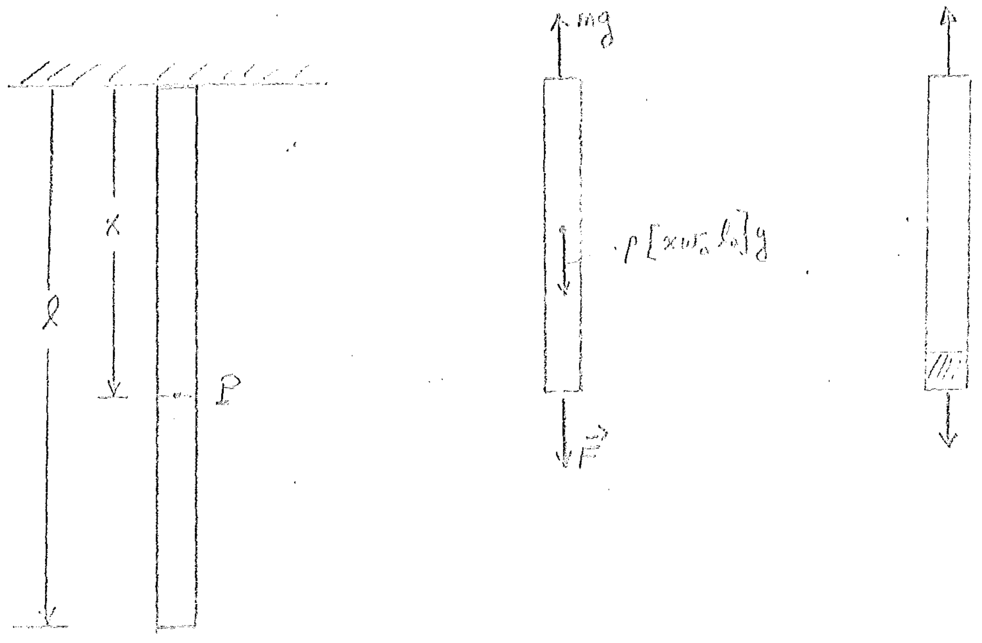
$$S_{xx} = \frac{F_x}{h_o w_o} = \frac{mg}{h_o w_o} - \rho g x$$

$$= \rho g\,[\ell_o - x] \qquad (1.9)$$

since $m = \rho\,[h_o w_o \ell_o]$. The stress component S_{xx} thus varies from point to point of the rod being a maximum at the top of the rod and zero at the bottom.

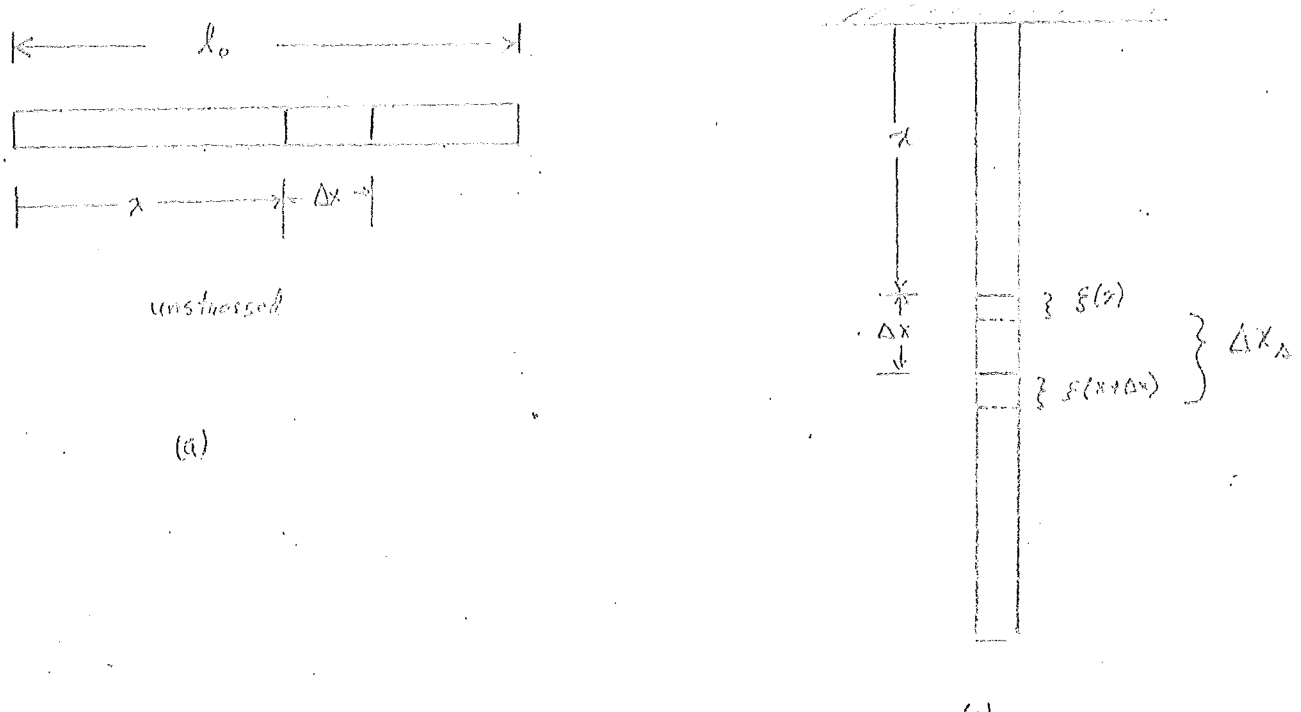
The strain at point P is defined in terms of the distortion undergone by a small segment, Δx, of the rod located at P in Fig. 1.12a. When the rod is hung from one end this segment is stretched to a length Δx_s as indicated in Fig. 1.12b. The strain (component) at P is defined as

$$\epsilon_{xx} = \lim_{\Delta x \to 0} \frac{\Delta x_s - \Delta x}{\Delta x}$$



(a) (b) (c)

Fig 1.11



(a) (b)

Fig 1.12

As depicted in Figures 1.12a and b, both the cross-section located at x and that at x + Δx are displaced slightly when the rod is suspended. The displacement that any given cross-section of the rod undergoes when the rod is hung depends on the location of the cross-section, and there is some, at the moment unknown, function, say $\xi(x)$ which specifies how far any given cross section is displaced. The displacements of the cross-sections at x and x + Δx are consequently labelled $\xi(x)$ and $\xi(x + \Delta x)$ respectively. It is evident from Fig. 1.12b that

$$\Delta x_s - \Delta x = \xi(x+\Delta x) - \xi(x)$$

so that

$$\epsilon_{xx} = \lim_{\Delta x \to 0} \frac{\xi(x+\Delta x) - \xi(x)}{\Delta x} = \frac{d\xi}{dx} \qquad (1,10)$$

The strain component ϵ_{xx} at a point is thus equal to the derivative of the function $\xi(x)$ which gives the displacement of each cross-section of the rod. It is generally assumed that the stress-strain relations expressed by equations (1.4) hold at every point. Consequently for the example we are considering

$$\epsilon_{xx} = \frac{1}{Y} S'_{xx} = \frac{1}{Y} \rho g (l_0 - x) \qquad (1.11)$$

Thus the strain also varies as x being a maximum at the supported end of the rod and zero at the bottom end. We can find $\xi(x)$ by integrating (1.11) obtaining

$$\xi(x) = \frac{1}{Y} \rho g \left[l_0 x - \frac{x^2}{2} \right] \qquad (1.12)$$

The constant of integration is zero in this instance since the top cross-section of the rod has zero displacement.

5. Thin beam

As a second example illustrating how one calculates stresses and strains let us consider a thin beam of length L resting on two knife edges and supporting a load W at its center, as indicated in Fig. 1.14a. For simplicity let us assume that the weight of the beam itself may be neglected. Let the beam have a rectangular cross-section of width w and height h. Let P be some general point in the rod, located a distance x from the left end and let us first consider the stresses at this point. (As mentioned earlier in a footnote, in calculating the stresses from equilibrium conditions one ignores any distortions that may have taken place when the beam was loaded.) Noting first that the entire beam is in equilibrium one concludes that the force exerted by each knife edge is W/2. Isolating the portion of the beam of length x as indicated in Fig. 1.13a, one notes that the forces acting on the isolated portion are the force of the knife edge at the left end and the forces exerted by the right hand portion of the rod. This latter set of forces are distributed in some manner over the cross-section of the beam as indicated in Fig. 1.13b. As far as equilibrium of the isolated portion is concerned, this set of distributed forces can be replaced by a single force $\vec{F}$ and a couple of moment M as indicated in Fig. 1.14c.* $\vec{F}$ in turn is usually resolved into two components F_x and F_y, referred to respectively as the normal and shearing forces. M is called the bending moment and is usually depicted as indicated in Fig. 1.14d. (More properly, M is the

---

* The proof that one can always find a single force and a couple whose effect as far as equilibrium is concerned is equivalent to an arbitrary set of forces, can be found in numerous texts on mechanics, e.g., Synge and Griffith, Principles of Mechanics (McGraw Hill, New York 1949) 2nd ed. p 50.

z-component of the torque, due to the couple, where the z-axis is taken to be perpendicular to the plane of Fig. 1.13a and pointing out of the paper.) From the fact that the isolated portion of the rod is also in equilibrium, it follows that

$$F_x = 0$$

$$F_y = - W/2 \tag{1.13}$$

$$M = Wx/2 \tag{1.14}$$

If we let the cross-section at P be the right hand surface of a small rectangular parallelepiped containing P, then this right hand surface is a positive surface and

$$S_{xx} = \frac{F_x}{wh} = 0$$

$$S_{yy} = \frac{F_y}{wh} = - \frac{W}{2wh}$$

The force components F_x, F_y and the couple M represent essentially the resultant or net effect of the set of distributed forces that the right hand portion of the rod exerts on the isolated portion. It turns out to be profitable to examine in more detail the nature of these distributed forces as revealed by an examination of the distortions undergone by the rod.

The deformation which the beam undergoes when loaded is shown greatly exaggerated in Fig. 1.14a. If the deformation is slight, it turns out that the center (dashed) line of the beam remains unchanged in length. Strips of the beam lying above this line are shortened, while strips lying below the line are lengthened. We isolate for consideration a small segment of the beam of length Δx, located a distance x from the left end. When the beam is deformed, the centerline of this small segment still has a length

Δx and lies some distance y below the centerline of the beam when the beam is unloaded. Fig. 1.14c is an enlarged view of the segment. The distance labelled R in this figure is the radius of curvature at the point of the dashed curve in Fig. 1.14b where Δx is located. The length of the shaded strip in Fig. 1.14c which lies a distance r below the centerline of the segment is $(R+r)\,\Delta\emptyset$. The length of this segment before the beam was loaded was $R\,\Delta\emptyset$, since with the beam unloaded all strips are the same length and the length of the center line doesn't change when the beam is deformed. The change in length of the shaded strip due to the deformation is thus $r\,\Delta\emptyset$ and consequently the strain (component) ϵ_{xx} at the point where the strip is located is $r\,\Delta\emptyset/R\,\Delta\emptyset$ or r/R. Since the strain at a point is related to the stress at a point by equation (1.4) the stress at the point where the shaded strip is located must be $Y\epsilon_{xx} = Yr/R$ To produce such a stress the actual forces $d\vec{F}$ exerted on the end surface of the shaded strip (see Fig. 1.14d) by the portion of the beam to the right must have a component dF_x, where

$$\frac{dF_x}{w\,dr} = Y\frac{r}{R} \quad\Longrightarrow\quad dF_x = Y\frac{r}{R}\,w\,dr$$

For a strip located a distance r above the centerline, the same considerations lead to the conclusion that the forces $d\vec{F}'$ on its end face must have a component dF'_x equal to $-\,dF_x$ as suggested in Fig. 1.15e. Both $d\vec{F}$ and $d\vec{F}'$ tend to rotate the element about the z-axis, the torque due to both being

$$d\tau_z = 2r\,dF_x = 2\frac{Yr^2}{R}\,w\,dr$$

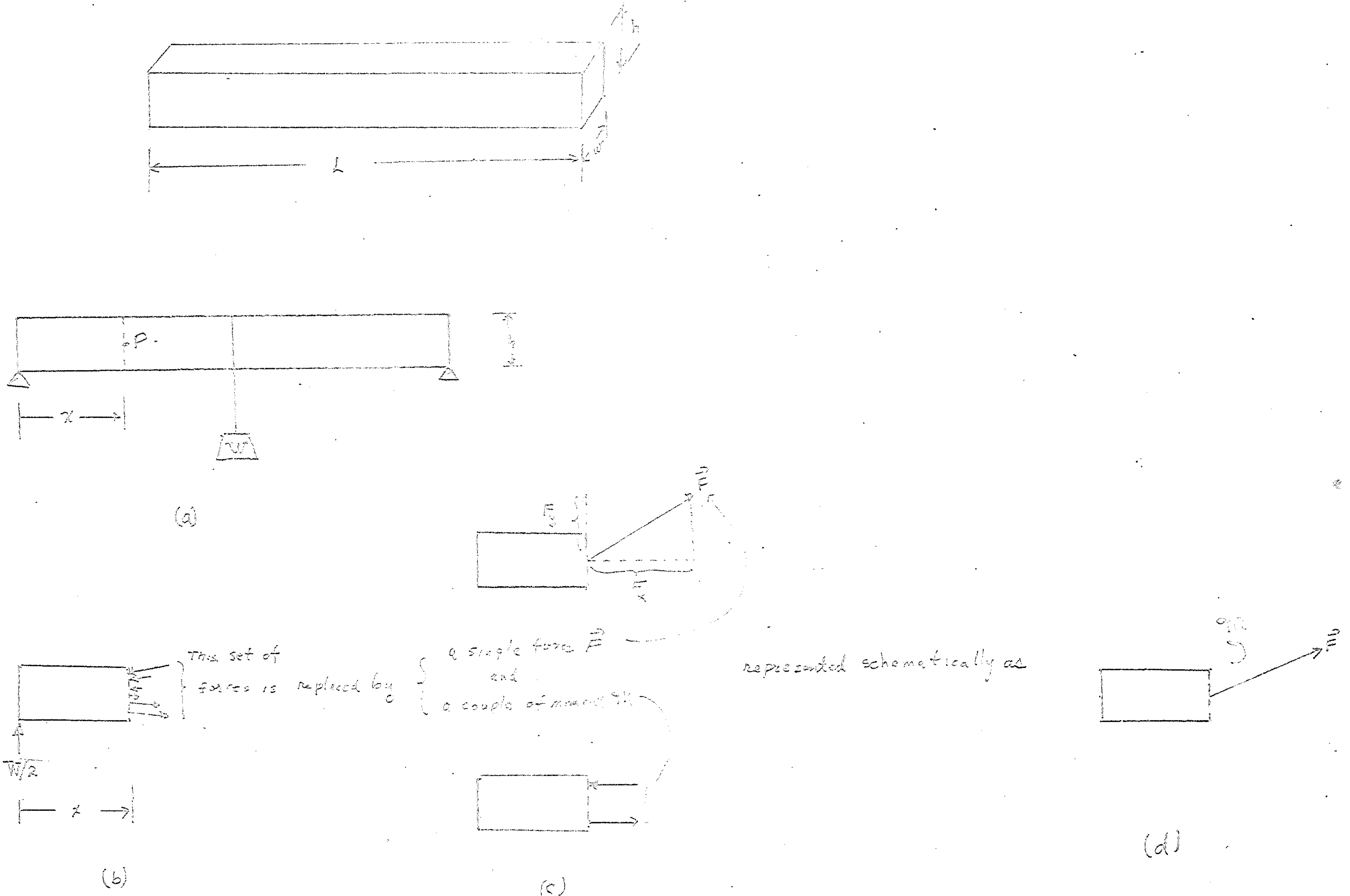

h
w
L
P
x
W
(a)
W/2
x
(b)
This set of forces is replaced by
a single force $\vec{F}$ and a couple of moment 𝔐
F_y
$\vec{F}$
F_x
(c)
represented schematically as
𝔐
$\vec{F}$
(d)

Fig 1.13

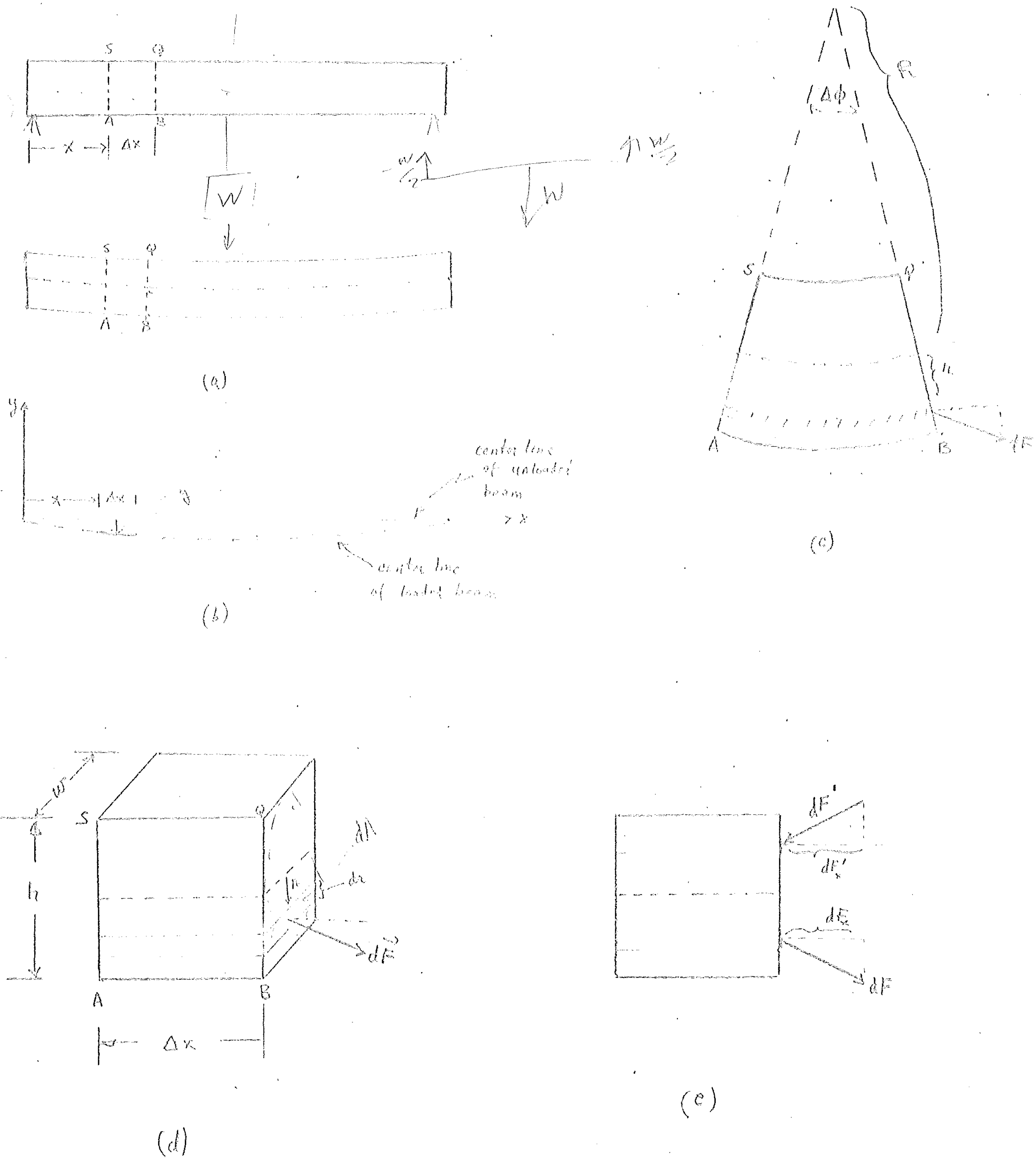

S
Q
A
B
x
Δx
W
W/2
W/2
W
(a)
y
x
Δx
y
x
center line of unloaded beam
center line of loaded beam
(b)
R
Δφ
S
Q
h
A
B
dF
(c)
w
S
Q
h
dA
dr
h
dF
A
B
Δx
(d)
dF'
dF'_x
dF_x
dF
(e)

Fig 1.14

The total torque due to forces acting on the end faces of all the strips is then the bending moment M. Thus

$$M = \int_0^{h/2} 2 \frac{Y h^2}{R} \omega \, dr = \frac{Y w h^3}{12 R}$$

This last expression relates the bending moment at a point to the radius of curvature of the rod at that point. In practically all textbooks on calculus it is shown that for any curve y(x), the radius of curvature at a point is given by

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Applying this relation to the curve of the centerline and remembering that for slight bending the slope $\frac{dy}{dx}$ at any point is small compared to unity, we have to a good approximation

$$R \cong \frac{1}{d^2y/dx^2}$$

so that the bending moment is given by

$$M = \frac{Y w h^3}{12} \frac{d^2y}{dx^2} \qquad (1.15)$$

This will prove to be a very useful and necessary relation later on in the derivation of the wave equation for waves in rods. We can use it now to find the curve into which the beam is bent when the load is applied. Substituting from (1.14) one obtains

$$\frac{d^2y}{dx^2} = \frac{12}{Y w h^3} \frac{W}{2} x$$

Integrating twice yields

$$y = \frac{W}{Y wh^3} x^3 + Cx + C^1$$

where C and C^1 are constants of integration. Taking y = 0 at x = 0 and $\frac{dy}{dx} = 0$ at $x = \frac{L}{2}$, the above expression becomes

$$y = \frac{W}{Y w h^3}\left(x^3 - \frac{3}{4}L^2 x\right) \qquad (1.16)$$

1.6 Rod under torsion

As a final application of the stress strain relation we consider the experiment illustrated in Fig. 1.15a, in which a rod is clamped at one end, and a known torque τ_{ext} is applied to the other end by means of the two forces labelled F. Since the entire rod is in equilibrium, the clamp must exert on the rod forces which give rise to a torque equal and opposite to that exerted at the top end of the rod. If one isolates a section of the rod of length x, since it too is in equilibrium, the forces exerted by the top section on the isolated portion must give rise to a torque exactly equal in magnitude to τ_{ext} as indicated in Fig. 1.14b. We can determine the nature of the forces giving rise to this torque, by considering the distortions that occur when the torque is applied.

When the rod is stressed by applying equal and opposite torques to the two ends, the rod undergoes a deformation in which each cross-section of the rod rotates about the ~~axes~~ axis of the rod through some angle which depends on where the cross-section is located. The angle through which a given cross section is rotated is measured between a line fixed in the cross section and a line fixed in space. For example, in Fig. 1.15, the line fixed in space is the ~~y-axes~~ y-axis, and the figure shows the top surface of the rod as having been rotated through an angle ϕ, and the cross section at x as being rotated through an angle $\psi(x)$. It is

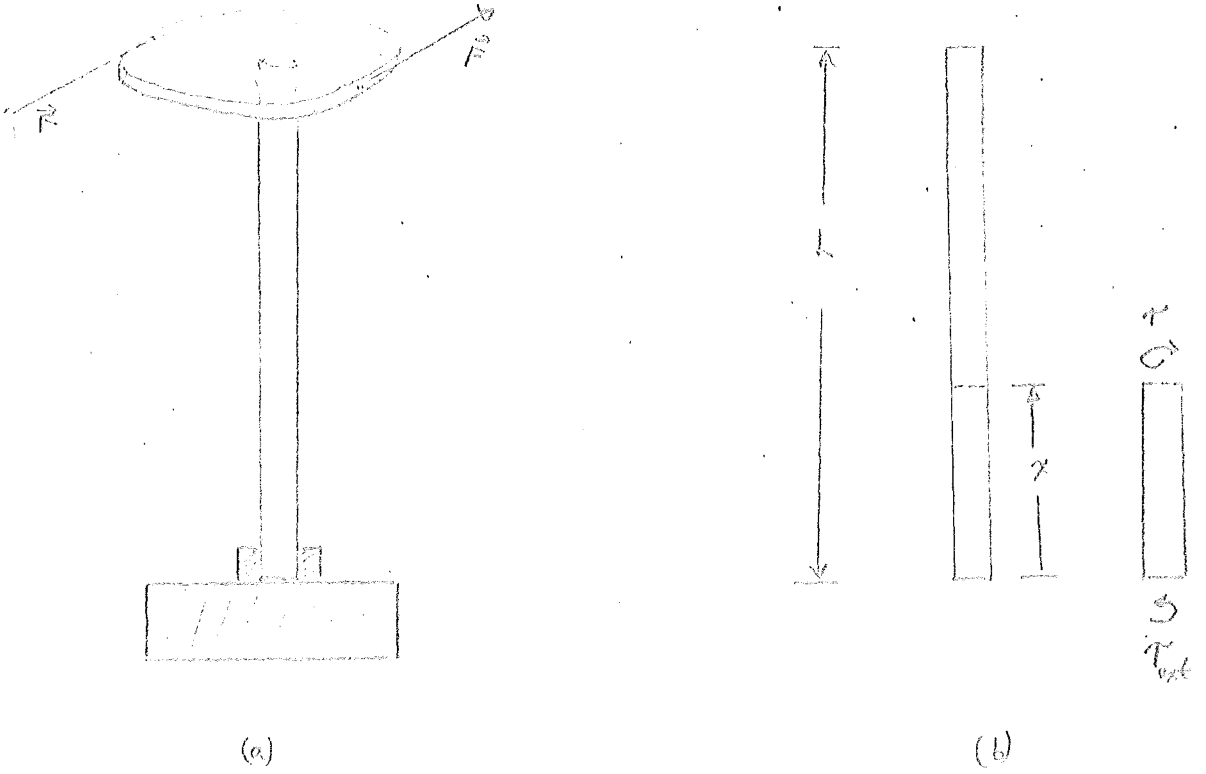


Fig 1.14

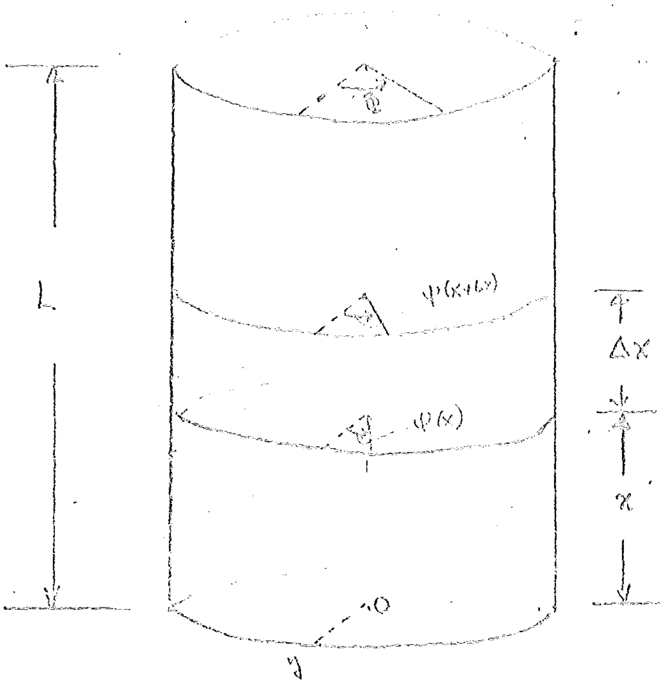


Fig 1.15

assumed that the bottom surface is prevented from rotating by the clamp. We isolate the section of the rod of length Δx and imagine it to be made up of a large number of thin concentric cylindrical shells. Fig. 1.16a shows one of these shells <u>before</u> the distortion has occurred. If the shell is thin the portion abcdefgh of the shell bounded by two radial sections making a small angle with each other, will be (very nearly) a rectangular parallelepiped. An enlarged view of this parallelepiped is shown in Fig. 1.16b. When the torque is applied, <u>each</u> radially line in the cross section at $x + \Delta x$ rotates through some angle labelled $\psi(x + \Delta x)$ while each radial line in the cross-section at x is rotated through an angle $\psi(x)$, as suggested in Fig. 1.16c. The effect of these two rotations on the rectangular parallelepiped is shown in Fig. 1.16d, where the bottom surfaces of undistorted and distorted parallelepiped are shown superimposed. It should be evident, that the effect is to produce a shearing strain θ equal to

$$\theta = \frac{dd'}{\Delta x} = \frac{r\left\{\psi(x+\Delta x) - \psi(x)\right\}}{\Delta x}$$

which in the limit as $\Delta x \rightarrow 0$ becomes

$$\theta = r\,\frac{d\psi}{dx} \qquad 1.17$$

Since the shearing strain and shearing stress are related by equation (1.7), there must exist at this point a shearing stress, $G\theta$, where G is the shear modulus. To produce such a shearing stress requires a set of forces $\overrightarrow{dF}$ acting tangentially to the top surface of the rectangular parallelepiped as indicated in Fig. 1.17 a and b. Such a set of forces would produce a torque of magnitude

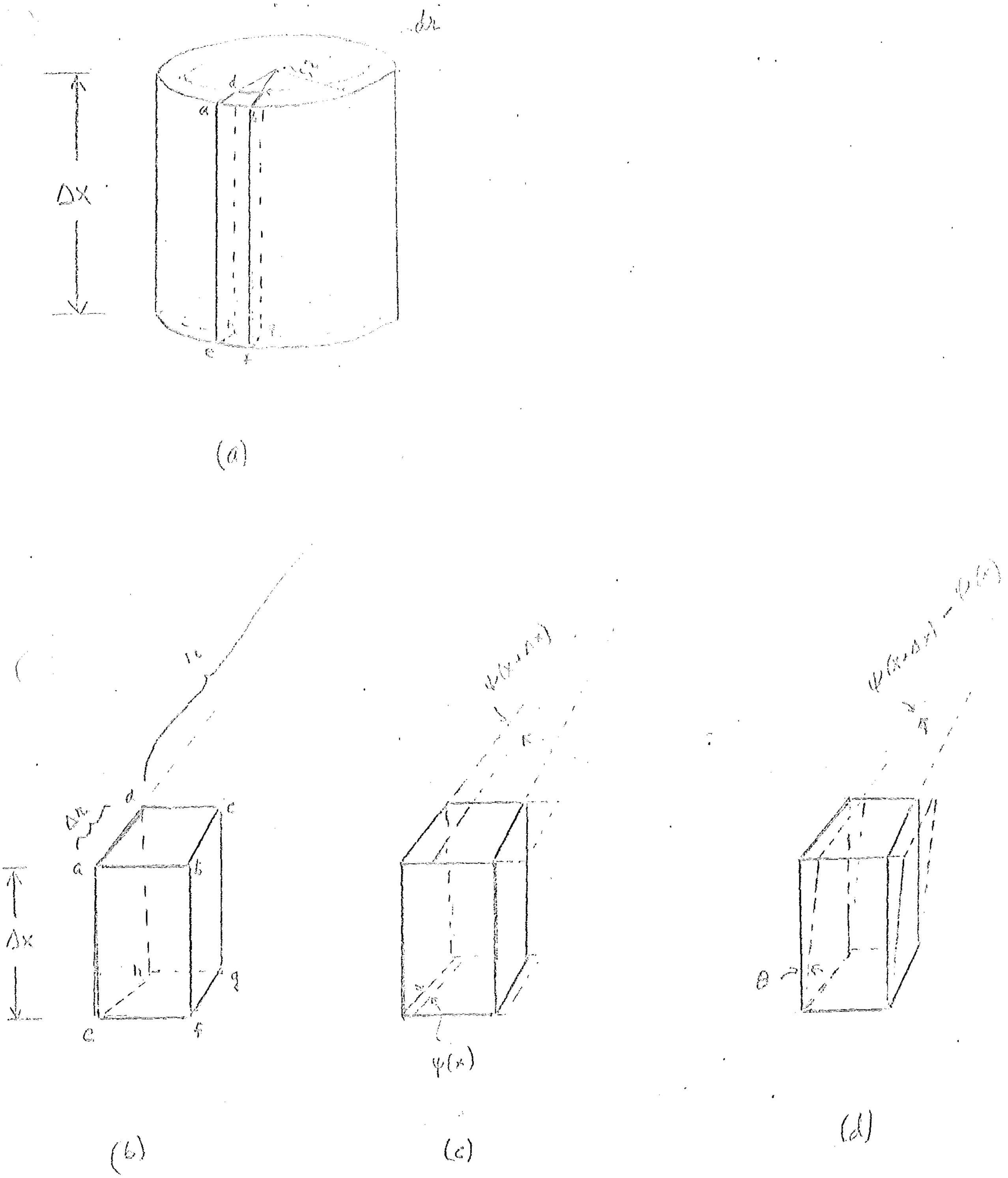


Fig 1.16

$$d\tau = r\,dF = r\,G\theta\,dA = r\left[G\,r\,\frac{d\psi}{dx}\right]dA$$

where dA is the area of the top face of the parallelepiped. Since all of the elements of the area of the top surface of the cylindrical shell have similar shearing forces, the total torque due to the forces acting on all the elements is

$$\Delta\tau = r\left[G\,r\,\frac{d\psi}{dx}\right]2\pi r\,\Delta r \qquad 1.18$$

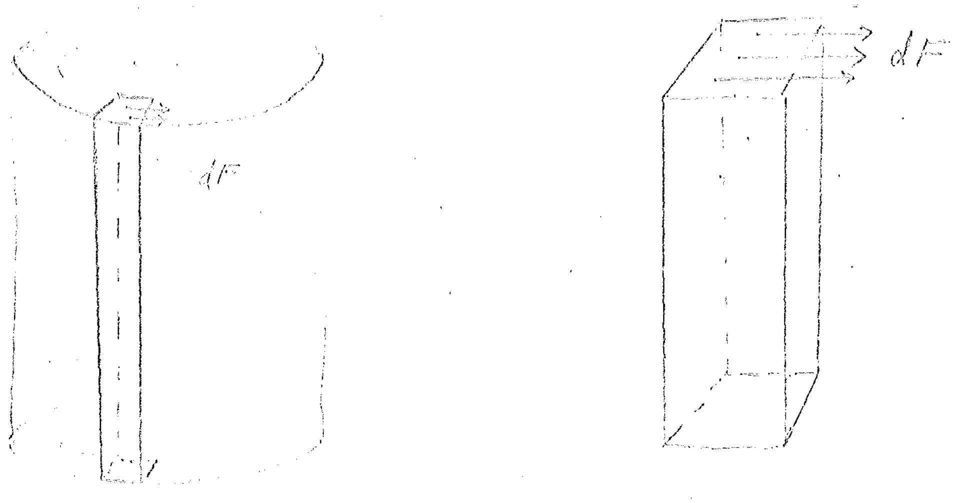
Since the isolated section of rod was considered to be made up of thin cylindrical shells, and since (1.18) applies to each shell, the total torque due to all the forces exerted on the surface at x by the portion of the rod above it is

$$\tau = \int_0^a r\,G\,r\,\frac{\partial\psi}{\partial x}\,2\pi r\,dr = \frac{G\,a^4\pi}{2}\,\frac{d\psi}{dx} \qquad (1.19)$$

where ~~R~~ a is the radius of the rod. This is an important relationship which will be useful later in the study of torsional waves in rods. From our consideration of equilibrium, the torque due to the forces exerted by one portion of the rod on the adjacent portion at any cross-section was equal to the externally applied torque T_{ext}. Consequently, the right hand side of (1.19) must equal T_{ext}, a constant. It follows that $\frac{d\psi}{dx}$ must also be constant so that

$$\psi = Cx + C'$$

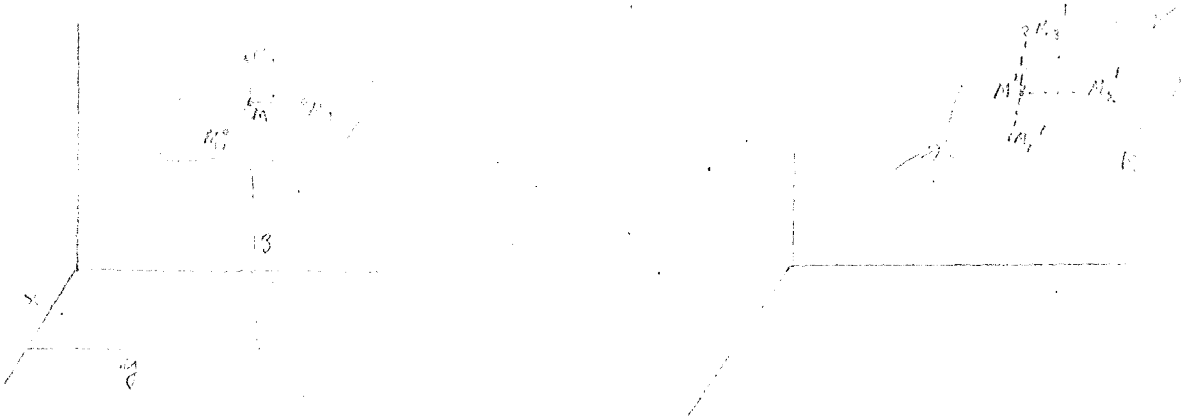
where C and C' are constants of integration. Noting that $\psi = 0$ when x = 0 and $\psi = \Phi$ when x = L, one obtains



(a)

(b)

Fig 1.17



(a)

(b)

Fig 1.18

Pure translation

(a)

(b) Pure rotation

Fig 1.19

$$\psi = \frac{\Phi}{L} x$$

The external torque required to twist one end of a rod through an angle Φ is thus

$$\tau_{st} = \frac{G a^4 \pi}{2} \frac{\Phi}{L} \qquad 1.20$$

Since $\frac{d\psi}{dx}$ is a constant, the shearing strain θ given by equation (1.17) is independent of x but does vary with r being a maximum for those elements located at the edge of the rod.

1.7 Generalized Concept of Strain

Let $M(x,y,z)$ be a point in the interior of an unstressed body (Fig. 1.18a). Imagine an observer at M has some means of identifying all of the points in his immediate neighborhood. Using three appropriate points, say M_1, M_2, M_3 he sets up a rectangular coordinate system with its origin at M such that $\overline{MM_1}$, $\overline{MM_2}$, $\overline{MM_3}$ correspond respectively to his x, y and z axes. If external forces are applied to the body (Fig. 1.18b) points M, M_1, M_2, and M_3 will in general be displaced to new positions, say M', M_1', M_2', and M_3'. If after this displacement, the observer reports that his coordinate system (determined by $\overline{M'M_1'}$, $\overline{M'M_2'}$, $\overline{M'M_3'}$) is still rectangular and all the neighboring points are precisely in the same positions relative to it as before the displacement, one says that the strain at M is zero. If the relative positions of the neighboring points has changed, then one says that there is a strain at M. It follows from this concept that if as illustrated in Fig. 1.19a) a body undergoes at pure translation, i.e. a motion in which each point moves the same distance along a path that is parallel to a fixed line, the strain is zero. Also, if as illustrated in Fig. 1.19b, a body undergoes a pure (small) rotation, θ, about some axes, the strain is also zero.

Let $N(x+dx,\ y+dy,\ z+dz)$ be a point in the neighborhood of $M(x,y,z)$ when the body is unstressed. When the body is stressed, then in general both M and N are displaced as illustrated in Fig. 1.20 which shows a two dimensional version of the situation. Let the x, y, and z components of the displacement $\vec{\Delta}$ of point M be ξ , η and ζ respectively, and let the corresponding quantities for the displacement $\vec{\Delta}$ of point N be be ξ_N , η_N , and ζ_N . Now the displacement $\vec{\Delta}$ and its components depend on the location of the point M, i.e. ξ , η and ζ are all functions of x, y and z. Since N is near M one has from the calculus

$$d\xi = \xi_N - \xi = \frac{\partial \xi}{\partial x} dx + \frac{\partial \xi}{\partial y} dy + \frac{\partial \xi}{\partial z} dz$$

$$d\eta = \eta_N - \eta = \frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy + \frac{\partial \eta}{\partial z} dz$$

$$d\zeta = \zeta_N - \zeta = \frac{\partial \zeta}{\partial x} dx + \frac{\partial \zeta}{\partial y} dy + \frac{\partial \zeta}{\partial z} dz$$

where the partial derivatives of ξ, η, and ζ are evaluated at the point M(x,y,z). If these partial derivatives are known for point M one can calculate $d\xi$, $d\eta$ and $d\zeta$ for any point in the neighborhood of M, and thus determine if there is a strain at M. To determine the relation between these partial derivatives and the strain at the point, one considers the distortion undergone by a tiny cube located at M as indicated in Fig. 1.20a. All points of this cube are in the neighborhood of M. Suppose for example, the external forces produce a strain such that ξ, η and ζ are all zero, and all of the partial derivatives of these quantities except $\frac{\partial \xi}{\partial x}$ are zero. Under these conditions the cube is stretched (or compressed) in the direction as indicated in Fig. 1.20b, the change in the x-dimension of the cube divided by the original x-dimension being exactly $\partial\xi/\partial x$, which was defined earlier as ϵ_{xx}. Similarly by considering a distortion in which only $\partial\eta/\partial y$ or $\partial\zeta/\partial z$ is zero, one can see that $\partial\eta/\partial x = \epsilon_{yy}$ and $\frac{\partial \zeta}{\partial z} = \epsilon_{zz}$. If the distortion is such that only $\partial\xi/\partial y$ is different from zero and positive, then the cube is sheared through an angle $\theta_1 = \partial\xi/\partial y$ as indicated in Fig. 1.20c. If the distortion is such that only $\partial\eta/\partial x$ is different from zero, the cube is sheared through an angle $\theta_2 = \partial\eta/\partial x$ as indicated in Fig. 1.20d. If both $\partial\eta/\partial x$ and $\partial\xi/\partial y$ are different from zero, and all other derivatives are zero, then the cube is sheared through an angle $\theta_1 + \theta_2$ as indicated in Fig. 121 a,b and c which shows the distortion of the top (or bottom) face of the cube. From considerations such as these, one concludes that the following quantities are sufficient to describe the strain at a

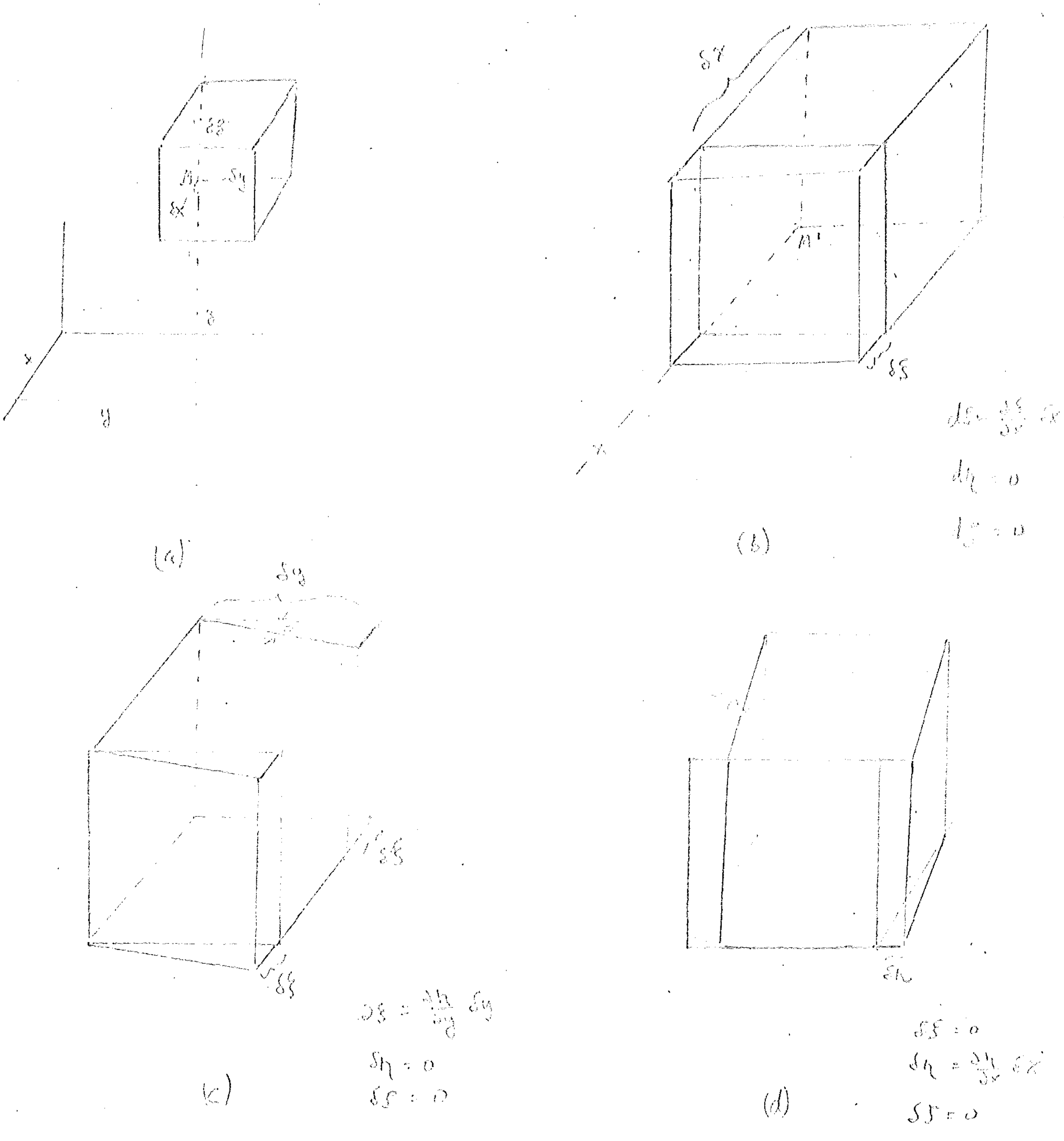


Fig 1.20

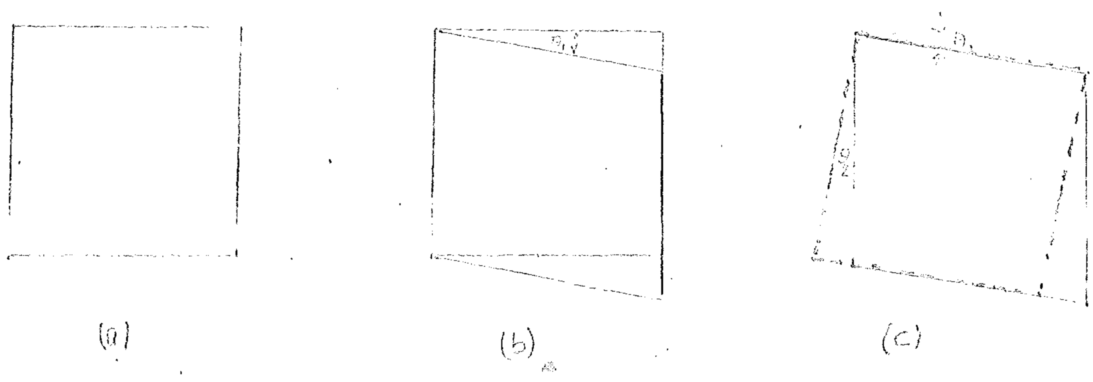


Fig 1.21

point*

$$\epsilon_{xx} = \frac{\partial \xi}{\partial x} \qquad \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) = \epsilon_{yz}$$

$$\epsilon_{yy} = \frac{\partial \eta}{\partial y} \qquad \epsilon_{xz} = \frac{1}{2}\left(\frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}\right) = \epsilon_{zx} \qquad (1.21)$$

$$\epsilon_{zz} = \frac{\partial \zeta}{\partial z} \qquad \epsilon_{yz} = \frac{1}{2}\left(\frac{\partial \eta}{\partial z} + \frac{\partial \zeta}{\partial y}\right) = \epsilon_{zy}$$

If all of the strain coefficients, ϵ_{xx}, ϵ_{xy}, ϵ_{x3}, ϵ_{yx}, ϵ_{yy}, ϵ_{yz} ϵ_{zx}, ϵ_{zy}, ϵ_{zz}, ~~and zz~~ are zero for point M, no distortion of the cube at M will occur. As indicated above, if ϵ_{xx}, ϵ_{yy}, or ϵ_{zz} are different from zero, the distortion consists of stretching or shortening the x,y or z dimensions of the cube, while if ϵ_{xy}, ϵ_{xz} or ϵ_{yz} are different from zero, the distortion consists of shearing the cube. The nine components ϵ_{xx}, ϵ_{xy} --- ---, ϵ_{zz}, only six of which are independent from what is called the strain tensor.

1.8 Generalized Concept of Stress, Stress Strain Relations

The stress at a point M(xmy,z) in a stressed body is defined in terms of the forces exerted on the three positive faces of a tiny cube located at point M as indicated in Fig. 122. Under conditions of equilibrium it is assumed that if the cube is sufficiently small, the forces exerted <u>on</u> any face of the cube by the material outside the cube are exactly equal and opposite to those forces exerted on the <u>opposite</u> face by the material outside

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* The 1/2 used in the definitions of ϵ_{xy}, ϵ_{xz} and ϵ_{yz} is arbitrary, and some authors omit this factor.

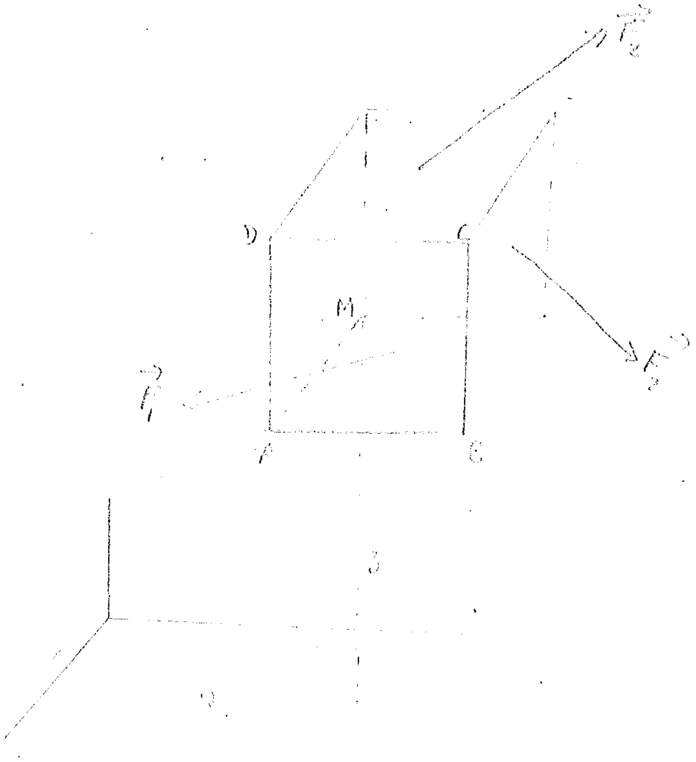

Fig 1.22

the cube, so that the forces exerted on the positive faces of the cube are actually representative of the forces exerted on the parallel surfaces passing through point M. If F_{xx}, F_{xy}, F_{xz} are the x, y and z components respectively on the force F_1 acting on face ABCD, F_{yx}, F_{yy}, F_{yz} the corresponding components of F_2, and F_{zx}, F_{zy}, F_{zz} the components of F_3, then the stress at M is specified by the nine components

$$S_{xx} = \frac{F_{xx}}{A} \qquad S_{xy} = \frac{F_{xy}}{A} \qquad S_{xz} = \frac{F_{xz}}{A}$$

$$S_{yy} = \frac{F_{yy}}{A} \qquad S_{yx} = \frac{F_{yx}}{A} \qquad S_{yz} = \frac{F_{yz}}{A}$$

$$S_{zz} = \frac{F_{zz}}{A} \qquad S_{zx} = \frac{F_{zx}}{A} \qquad S_{zy} = \frac{F_{zy}}{A}$$

where $A = dx\ dy\ dz$ is the area of a face of the cube. For equilibrium of the cube as regards rotation one must have $S_{xy} = S_{yx}$, $S_{xz} = S_{zx}$ and $S_{yz} = S_{zy}$ so there are actually on six independent stress components. The components S_{xx}, S_{yy} and S_{zz} are called normal stresses, while the other components are called shearing stresses.

It is generally assumed that each stress component is a linear function of six strain components*, i.e.

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* One could equally well assume that each strain component is a linear function of the six stress components.

$$S_{xx} = C_{11}\epsilon_{xx} + C_{12}\epsilon_{yy} + C_{13}\epsilon_{zz} + C_{14}\epsilon_{xy} + C_{15}\epsilon_{xz} + C_{16}\epsilon_{yz}$$

$$S_{yy} = C_{21}\epsilon_{xx} + C_{22}\epsilon_{yy} + C_{23}\epsilon_{zz} + C_{24}\epsilon_{xy} + C_{25}\epsilon_{xz} + C_{26}\epsilon_{yz}$$

$$\vdots$$

$$S_{yz} = C_{61}\epsilon_{xx} + C_{62}\epsilon_{yy} + C_{63}\epsilon_{zz} + C_{64}\epsilon_{xy} + C_{65}\epsilon_{xy} + C_{66}\epsilon_{yz}$$

where the coefficients C_{11}, C_{12}, ... C_{65}, C_{66} are constants, characteristic of the material. As one might suspect, for an <u>isotropic</u> solid some of these coefficients are zero and many of the others are equal; in fact it turns out that there are only <u>two</u> independent coefficients. For an isotropic solid the strain relations become

$$\left.\begin{aligned} S_{xx} &= (C_1 + C_2)\epsilon_{xx} + C_1\epsilon_{yy} + C_1\epsilon_{zz} \\ S_{yy} &= C_1\epsilon_{xx} + (C_1 + C_2)\epsilon_{yy} + C_1\epsilon_{zz} \\ S_{zz} &= C_1\epsilon_{xx} + C_1\epsilon_{yy} + (C_1 + C_2)\epsilon_{zz} \\ S_{xy} &= C_2\epsilon_{xy} \\ S_{xz} &= C_2\epsilon_{xz} \\ S_{yz} &= C_2\epsilon_{yz} \end{aligned}\right\} \quad \text{HOMOGENEOUS ISOTROPIC SOLIDS} \tag{1.22}$$

where

$$C_1 = \frac{\sigma Y}{(1+\sigma)(1-2\sigma)} \quad \text{and} \quad C_2 = \frac{Y}{1+\sigma}$$

Here Y is Young's modulus and σ is Poisson's ratio. The first three of these equations are, of course, the inverse of equations (1.4). It is worth mentioning again that $\epsilon_{xx}, \epsilon_{xy} \ldots$ in (1.22) can be interpreted as the strains resulting from <u>changes</u> in the stresses of amounts S_{xx}, S_{xy} .... .

For an ideal fluid, the stress strain relationships are even simpler:

$$S_{xx} = S_{yy} = S_{zz} = B(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}) \qquad (1.23)$$

FOR THE IDEAL FLUID

$$S_{xy} = S_{xz} = S_{yz} = 0$$

where B is the bulk modulus. Also for a fluid, the change in the stress is simply equal to the change the negative of the change in pressure, $\triangle P$, so that

$$\triangle P = -B(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$$

or

$$\triangle P = -B(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}) \qquad (1.24)$$

This relationship will prove useful in developing the wave equation for waves in fluids.

CHAPTER II HARMONIC MOTION

Simple harmonic motion (along with uniform circular motion) is perhaps the simplest type of repetitive motion that one can imagine. Partly because of this, and partly because of the simplicity of its mathematical representation, simple harmonic motion proves to be useful in the description of a great many diverse physical phenomena. It plays a particularly important role in the study of vibrations and waves; as we shall learn presently, the vibrations of any material object or any small portion of a medium through which a wave is travelling is almost invariably assumed to be simple harmonic or made up of some combination of simple harmonic motions. Because of its importance, it will be worthwhile to review harmonic motion before beginning the study of waves.

2.1 The Simple Harmonic Oscillator

Consider as depicted in Fig. 1.1 the simplest possible case: a particle of mass m supported by a horizontal frictionless surface and subjected to a restoring force supplied by a massless spring of force constant K. If x is the displacement of the mass from its equilibrium position, Newton's second law applied to the mass yields

$$-Kx = m\ddot{x}$$

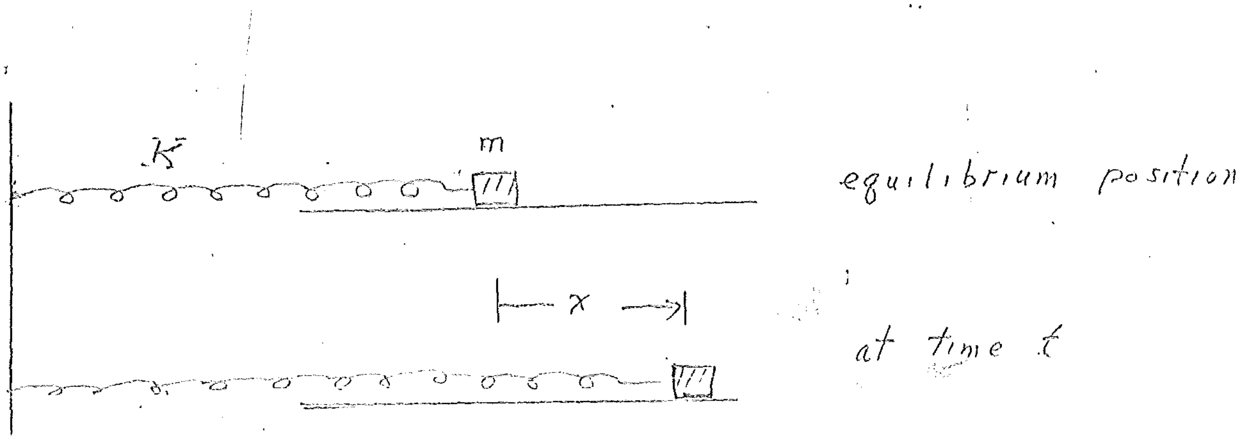
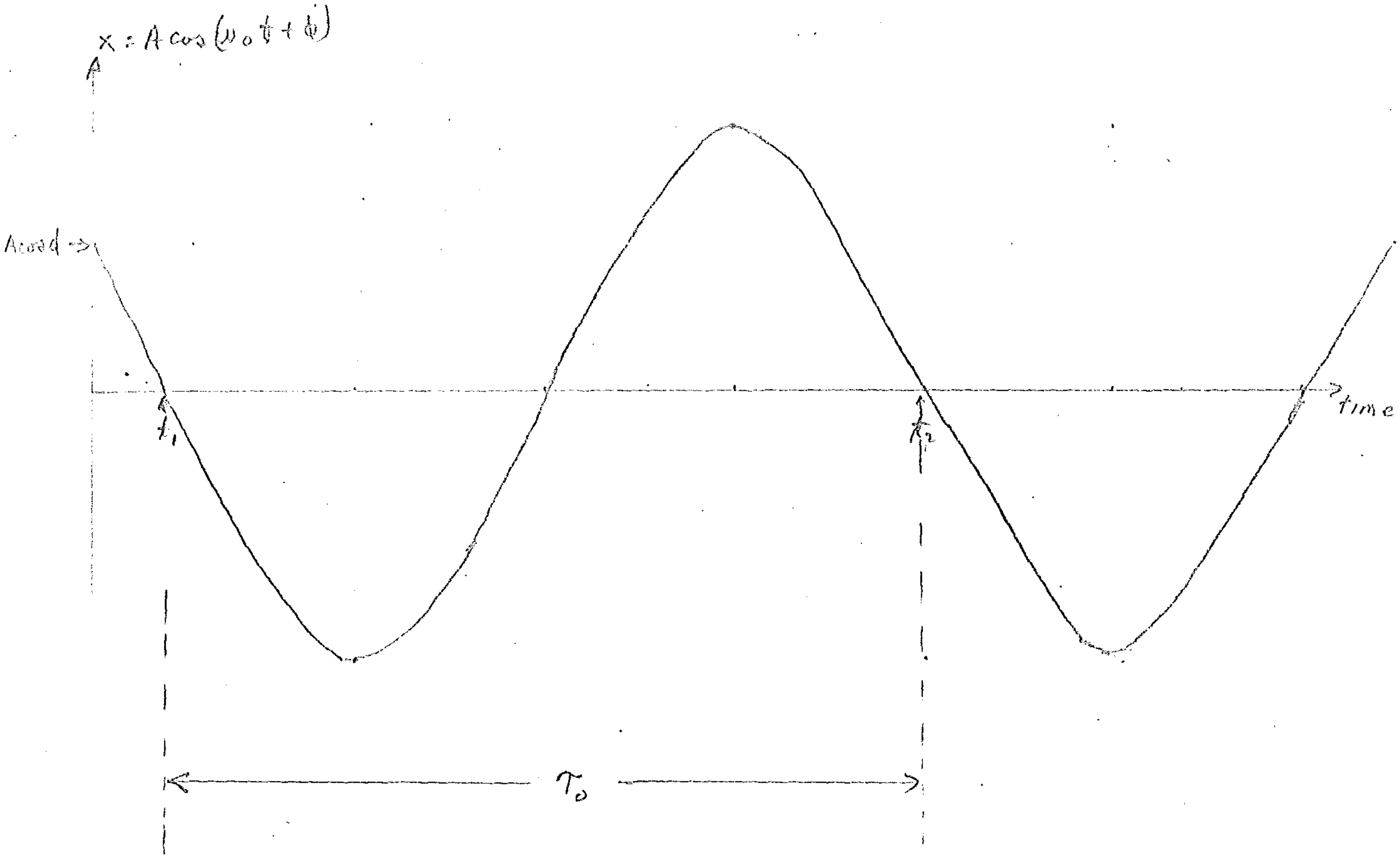


Fig 2.1



$$t_1 = \frac{\pi/2 - \phi}{\omega_0} \qquad t_2 = \frac{5/2\,\pi - \phi}{\omega_0}$$

Fig 2.2

where the x stands for $\frac{d^2x}{dt^2}$. This differential equation is called the equation of motion of the particle. Our task is to find a solution of this differential equation, since we know that any function $x(t)$ which describes how the particle moves must be a solution of the equation of motion, Fundamentally finding a solution of a differential equation is a process of trial and error. There are, however, some general methods of finding solutions of differential equations which are successful in many instances and we will use one of these general methods to find a solution. For convenience let

$$\omega_0 = \sqrt{K/m} \tag{2.1}$$

so that the equation of motion may be written

$$\ddot{x} + \omega_0^2 x = 0 \tag{2.2}$$

The general method consists of guessing that there is a solution of the form

$$x(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots. \tag{2.3}$$

where a_0, a_1, a_2.... are all constants. If such a solution exists then

$$\ddot{x}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 \ldots$$

Substituting this expression along with (2.3) into equation (2.2) one obtains

$$[2a_2 + \omega_0^2 a_0] + [6a_3 + \omega_0^2 a_1]t + [12a_4 + \omega_0^2 a_2]t^2 + [20a_5 + \omega_0^2 a_3]t^3 + \ldots = 0$$

For (2.3) to be a solution of (2.2) the above expression must be identically zero, i.e., zero for all possible values of time. This condition would obviously be satisfied if each of the bracketed quantities were equal to zero. If a_0 and a_1 are given arbitrary values, then the first bracket can be made zero by choosing

$$a_2 = -\frac{\omega_0^2}{2} a_0$$

the second bracket by choosing

$$a_3 = -\frac{\omega_0}{6} a_1$$

the third bracket by choosing

$$a_4 = -\frac{\omega_0^2}{12} a_2 = \frac{\omega_0^4}{24} a_0$$

and so on. Thus (2.3) will be a solution of the equation of motion for arbitrarily chosen values of a_0 and a_1 provided the other coefficients have the values determined as indicated above. Substituting these values in (2.3) one obtains after rearranging the following solution of the equation of motion

$$X(t) = a_0 [1 + \frac{(\omega_0 t)^2}{2!} + \frac{(\omega_0 t)^4}{4!} - \frac{(\omega_0 t)^6}{6!} + \ldots]$$

$$+ \frac{a_1}{\omega_0} [\omega_0 t - \frac{1}{3!}(\omega_0 t)^3 + \frac{1}{5!}(\omega_0 t)^5 + \ldots]$$

The infinite series contained in the first bracket is a Taylor's expansion for $\cos \omega_0 t$, while that in the second bracket is an expansion for $\sin \omega_0 t$. The solution can therefore be written in the more familiar form

$$X(t) = C \cos \omega_0 t + D \sin \omega_0 t \qquad (2.4)$$

where C and D have been used to replace a_1 and a_1/ω_0 respectively.

In the expression (2.4), C and D are arbitrary in the sense that (2.4) is a solution of the equation of motion no matter what values are assigned to them. Since the equation of motion is a <u>second</u> order differential equation and since (2.4) has <u>two</u> arbitrary constants, it may be considered the <u>general</u> solution of the differential equation. If the position and velocity of the particle are specified at some instant of time, then these so-called initial conditions determine particular values of C and D and the resulting solution is said to be a <u>particular</u> solution of the differential equation. For example, $x = 3 \cos \omega_0 t$ is a particular solution of (2.2) corresponding to releasing the mass m from rest at a distance 3 units from its equilibrium position.

For any arbitrarily chosen values of C and D it is always possible to find a number A and an angle ϕ such that $C = A \cos \phi$ and $D = - A \sin \phi$. The solution (2.4) can therefore be written in the alternate form

$$x = A \cos(\omega_0 t + \phi) \tag{2.5}$$

A plot showing the x - coordinate of the particle as a function of time is shown in Fig. 2.2. It should be noted that the motion repeats itself after a time interval

$$\tau_0 = 2\pi/\omega_0 = 2\pi \sqrt{\frac{m}{K}}$$

This time interval is called the period of the motion, and its reciprocal

$$f_0 = \frac{1}{\tau_0} = \frac{1}{2\pi} \sqrt{K/m}$$

is called the frequency. The quantity ω_0 is also loosely referred to as the frequency although the term "angular" frequency would perhaps be more suitable. The magnitude of the largest displacement of the particle from its equilibrium position is called the amplitude of the motion. It corresponds to the absolute value of A in equation (2.5).

2.2 Complex Form of Solution

One can obtain any number of particular solutions of (2.2) by simply inserting different values of A and ϕ into (2.5). Let $x_1(t)$ and $x_2(t)$ be two of these particular solutions. Since they are both solutions we must have

$$\ddot{x}_1 + \omega_0^2 x_1 = 0$$

and

$$\ddot{x}_2 + \omega_0^2 x_2 = 0$$

If the second of these is multiplied by $i = \sqrt{-1}$ and added to the first, one obtains

$$\ddot{x}_1 + i\ddot{x}_2 + \omega_0^2 (x_1 + i x_2) = 0 \tag{2.6}$$

Let $\underset{\sim}{x}(t)$ be defined as follows*

$$\underset{\sim}{x}(t) = x_1(t) + i x_2(t) \tag{2.7}$$

Functions like $\underset{\sim}{x}(t)$ which consist of this simple arrangement of two real functions form a special class# of complex functions. Differentiation or integration of this special class of functions

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*A wavy line underneath a symbol indicates the symbol stands for a complex quantity.

#All complex functions encountered in this book are of this special class.

is accomplished by treating $\underset{\sim}{i}$ as if it were a real constant. Thus

$$\dot{\underset{\sim}{X}}(t) = \dot{X}_1(t) + \underset{\sim}{i}\,\dot{X}_2(t)$$

$$\ddot{\underset{\sim}{X}}(t) = \ddot{X}_1(t) + \underset{\sim}{i}\,\ddot{X}_2(t)$$

$$\int \underset{\sim}{X}(t)\,dt = \int X_1(t)\,dt + \underset{\sim}{i} \int X_2(t)\,dt$$

From these rules, it is possible to write (2.6) as

$$\ddot{\underset{\sim}{X}} + \omega_0^2 \underset{\sim}{X} = 0 \qquad (2.8)$$

This complex differential equation is identical in form to (2.2). A solution of this complex differential equation is any complex function of the form (2.7) which satisfies it. It can be easily shown if it is not already apparent that the function

$$\underset{\sim}{X}(t) = A\cos(\omega_0 t + \phi) + \underset{\sim}{i}\,A\sin(\omega_0 t + \phi) \qquad (2.9a)$$

is a solution of (2.8). Using Euler's theorem one can write this in the form

$$\underset{\sim}{X}(t) = \underset{\sim}{A}\,e^{\underset{\sim}{i}\,\omega_0 t} \qquad (2.9b)$$

where $\underset{\sim}{A} = Ae^{i\phi}$ is a complex number. Now the real part of (2.9a) or (2.9b) corresponds exactly to (2.5), the general solution of the equation of motion (2.2). For reasons that will become apparent later one, one prefers to work with (2.9b) and to regard it as the equation which describes the motion of the particle. It is, of course, the real part which actually describes the motion of the particle.

2.3 Velocity, Acceleration and Phase Relationships

Equation (2.5) gives the x coordinate of the mass m at any instant. The velocity and acceleration can be obtained by successive differentiations:

$$\dot{x} = -A\omega_0 \sin(\omega_0 t + \phi) \tag{2.10}$$

$$\ddot{x} = -A\omega_0^2 \cos(\omega_0 t + \phi) \tag{2.11}$$

Now x, $\dot{x}$, and $\ddot{x}$ all vary sinusoidally with the time, and all have precisely the same period. However, no two of the three quantities attain their largest (peak) positive values at exactly the same time. For example x attains its peak positive value, A, at times t' such that

$$\omega_0 t' + \phi = 0, 2\pi, 4\pi, \ldots.$$

At such times, $\dot{x}$ is zero, and $\ddot{x}$ is at its peak negative value. When two sinusoidally varying quantities having the same period attain their positive peak values at different times they are said to differ in phase, the phase difference bing defined as $2\pi(t_1-t_2)/\tau_0$ where t_1 is a time at which one of the quantities attains its maximum positive value, t_2 is the time nearest to t_1 at which the other quantity attains its maximum positive value, and τ is the period. The phase difference, thus defined, is in radians, although it is often expressed in degrees. Since x attains its peak positive value at times t" such that

$$\omega_0 t'' + \phi = 3/2\pi,\ 7/2\pi,\ 11/2\pi, \ldots$$

and $\ddot{x}$ its largest positive value at times t"' such that

$$\omega_0 t''' + \phi = \pi,\ 3\pi,\ 5\pi, \ldots$$

we can see that x differs in phase from $\dot{x}$ by $\pi/2$ radians or 90^o and from $\ddot{x}$ by π radians or 180^o.

If we use the complex exponential form, (2.9b), of the solution we have*

$$\underset{\sim}{x} = A\cos(\omega_0 t+\phi) + i A\sin(\omega_0 t+\phi) = \underset{\sim}{A} e^{i\omega_0 t}$$

$$\dot{\underset{\sim}{x}} = -\omega_0 A\sin(\omega_0 t+\phi) + i\omega_0 A\cos(\omega_0 t+\phi) = i\omega_0 \underset{\sim}{A} e^{i\omega_0 t} = i\omega_0 \underset{\sim}{x}$$

$$\ddot{\underset{\sim}{x}} = -\omega_0^2 A\cos(\omega_0 t+\phi) - i\omega_0^2 A\sin(\omega_0 t+\phi) = -\omega_0^2 \underset{\sim}{A} e^{i\omega_0 t} = -\omega_0^2 \underset{\sim}{x}$$

At any given instant of time $\underset{\sim}{x}$, $\dot{\underset{\sim}{x}}$, and $\ddot{\underset{\sim}{x}}$ are complex numbers and

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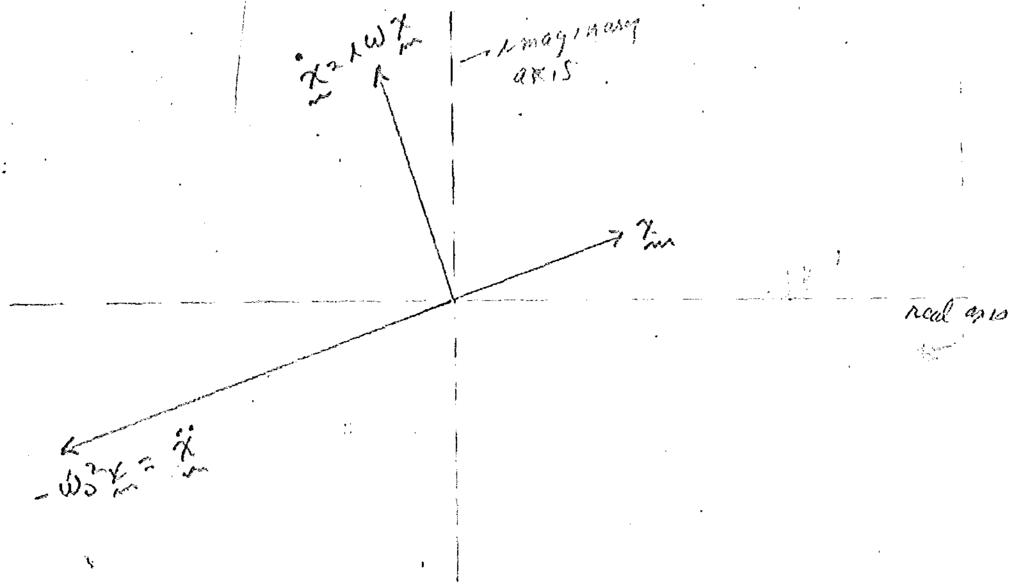
*Note that differentiating or integrating the function [illegible] is eactly equivalent to differentive or integrating $\underset{\sim}{A} e^{i\omega_0 t}$ treating $\underset{\sim}{A}$ and [illegible] as if they were real constants.

may be represented in the complex plane as shown in Fig. 2.3. Note, that although the position of $\underset{\sim}{x}$ is arbitrary, since it depends upon the particular instant of time chosen, once $\underset{\sim}{x}$ is drawn, the positions of $\dot{\underset{\sim}{x}}$ and $\ddot{\underset{\sim}{x}}$ are fixed, since $\dot{\underset{\sim}{X}} = i\omega_0 \underset{\sim}{X}$ and $\ddot{\underset{\sim}{X}} = -\omega_0 \underset{\sim}{X}$. Note further that the angle between $\dot{\underset{\sim}{x}}$ and $\underset{\sim}{x}$ is $\frac{\pi}{2}$ or 90°, precisely the phase difference between $\dot{x}$ and x while that between $\underset{\sim}{x}$ and $\ddot{\underset{\sim}{x}}$ is 180°, exactly the phase difference between x and $\ddot{x}$. It should thus be apparent that the phase relations between the various quantities are more readily deduced from the complex exponential form of the solution than from the real form. In Fig. 2.3. the projections of the vectors $\underset{\sim}{x}$, $\dot{\underset{\sim}{x}}$, and $\ddot{\underset{\sim}{x}}$ on the real axis are the real parts of these quantities and hence represent, respectively, the values of x, $\dot{x}$, and $\ddot{x}$ at this particular instant. As time increases the three vectors each rotate counterclockwise with an angular velocity ω_0. Because $\dot{\underset{\sim}{x}}$ is 90° <u>counterclockwise</u> from $\underset{\sim}{x}$ it is said to <u>lead</u> $\underset{\sim}{x}$ by $\frac{\pi}{2}$ or 90°. $\ddot{\underset{\sim}{x}}$ may be said to <u>lead</u> or <u>lag</u> $\underset{\sim}{x}$ by π radians or 180° since one ordinarily speaks of quantities leading or lagging by angles of π radians or less.

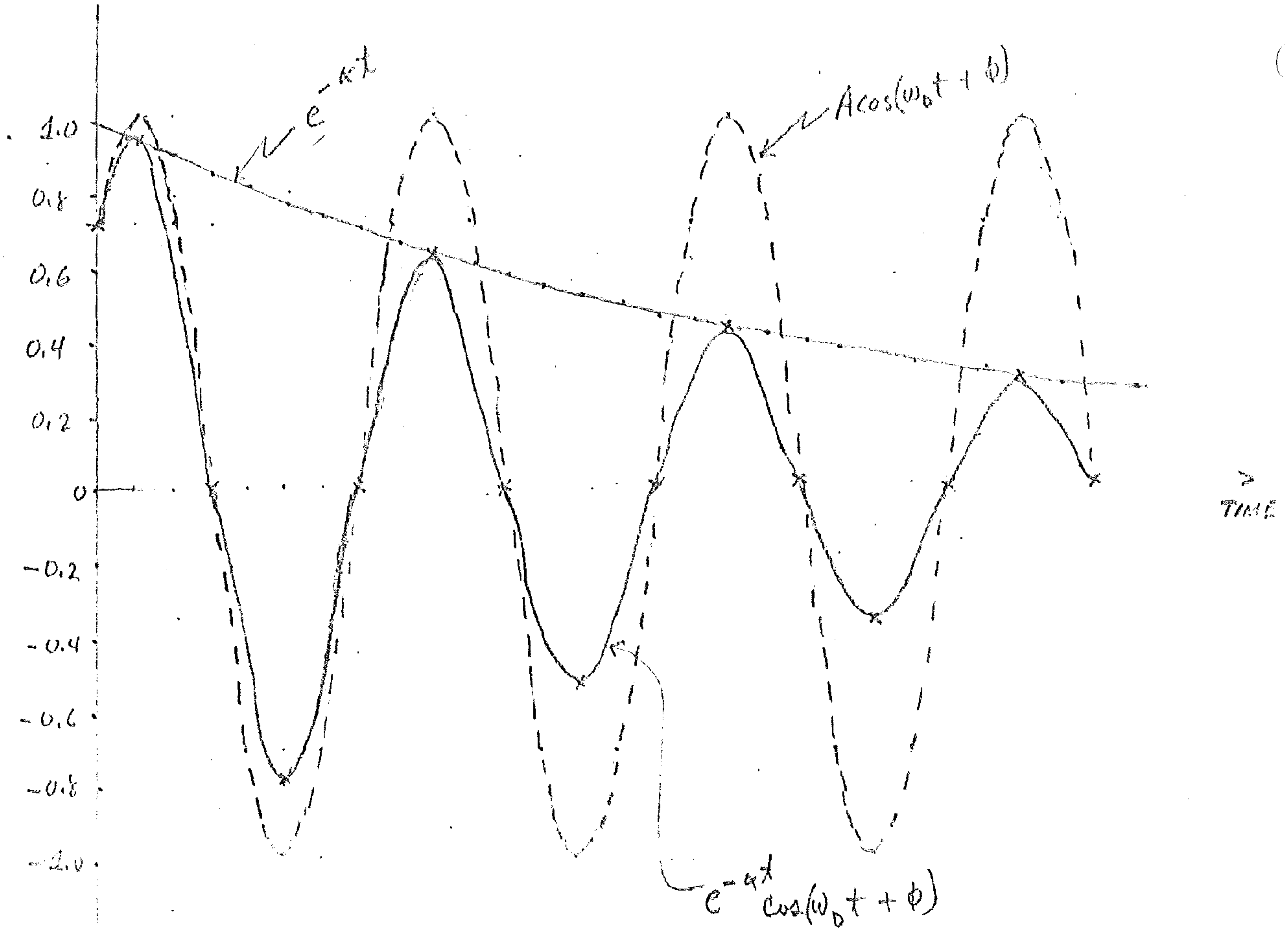
2.4 Energy of the Simple Harmonic Oscillator

The total mechanical energy E of the oscillator is the sum of its kinetic and potential energies. The kinetic energy by definition is $m\,\dot{x}^2/2$. The potential energy of a mass m in a given position may be defined as the work done <u>by</u> the conservative



$\dot{x} = i\omega x_m$
imaginary axis
x_m
real axis
$-\omega_0^2 x_m = \ddot{x}$

Fig 2.3



$e^{-\alpha t}$
$A\cos(\omega_0 t + \phi)$
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
TIME
$e^{-\alpha t}\cos(\omega_0 t + \phi)$

Fig 2.4

forces (in this case the spring force) as the mass is moved from the given position x to an arbitrarily chosen reference position (chosen for convenience in this case to coincide with the equilibrium position of the particle.) We have then by definition

$$V(x) = \int_x^0 -Kx\,dx = \frac{1}{2}Kx^2$$

for the potential energy. The total energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2$$

Substituting from (2.10) and (2.5) one obtains

$$E = \frac{1}{2}mA^2\omega_0^2 \sin^2(\omega_0 t + \phi) + \frac{1}{2}KA^2 \cos^2(\omega_0 t + \phi)$$

$$= \frac{1}{2}KA^2 = \frac{1}{2}mA^2\omega_0^2$$

The total energy is thus constant as we would expect since the only force acting is a conservative one.

2.5 Damped Harmonic Motion

From experience we have learned that there is no real oscillating system which corresponds exactly to a simple harmonic oscillator. All real oscillating systems are subject to dissipative forces, and if left to themselves (i.e. if no energy is supplied regularly from some outside source) the oscillations will eventually cease. To make our hypothetical oscillator correspond more closely to a real oscillating system, we need to include a dissipative or damping force. Conventionally one selects a dissipative force which is proportional to the velocity of the particle and is opposite in direction. This choice results in an equation of motion, the solution of which corresponds reasonably closely to the observed motion of certain real oscillating systems. The equation of motion with this damping force included becomes

$$m\ddot{x} + R\dot{x} + Kx = 0$$

For convenience let

$$\omega_0 = \sqrt{K/m} \quad ; \quad \alpha = R/2m$$

so that the equation of motion may be written

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0 \qquad (2.12)$$

It can be readily verified by differentiating and substituting in (2.12) that

$$x = e^{-\alpha t}\left[A\cos(\omega_0 t + \phi)\right] \qquad (2.13)$$

where

$$\omega_D = \sqrt{\omega_0^2 - \alpha^2}$$

is a solution of (2.12).* This will be found to be a solution for any arbitrarily chosen values of A and ϕ ; hence may be regarded as the general solution of (2.12). The quantity in brackets is exactly the same form as (2.5), the solution of the undamped oscillator. The type of motion represented by (2.13) is shown in Fig. 2.4 where the cosine term and the exponential term are sketched separately and multiplied at each point to obtain the value of x. It is seen that the motion is oscillatory with a gradually decaying amplitude. While strictly speaking this is not a periodic function, we may define the frequency as the number of times per second that the particle passes through its equilibrium position in the positive direction. The frequency is thus

$$f_D = \frac{\omega_D}{2\pi} = \frac{\sqrt{\omega_0^2 - \alpha^2}}{2\pi} = \frac{\sqrt{K/m - (R/2m)^2}}{2\pi} \qquad (2.14)$$

If R/2m is small compared to K/m, this frequency is only slightly smaller than the frequency of an undamped oscillator of the same mass and spring constant. If $\alpha = R/2m$ is small compared to 1, then over any short time interval, say $t_2 - t_1$, the term $Ae^{-\alpha t}$ is approximately constant, i.e. the values

$$Ae^{-\alpha t_1} \qquad Ae^{-\alpha t_2} \qquad Ae^{-\alpha \frac{t_1+t_2}{2}}$$

---

*There are three types of solutions of equation (2.12) depending on whether ω_0 is greater than, equal to, or less than α . The solution of most interest in our present discussion is (2.13) which is the solution when $\omega_0 > \alpha$

are all very nearly the same, and over this time interval the motion can be considered undamped harmonic motion with an amplitude $Ae^{-\alpha t_1}$ (or either of the other two values). In this sense we can say that when $\alpha << 1$, the amplitude at any time t can be considered to be $Ae^{-\alpha t}$. It follows from this that $\frac{1}{\alpha} = \frac{2m}{R}$ is the time for the amplitude to decrease to $\frac{1}{e}$ of its initial value. By measuring this time one can determine α. If α is not small compared to 1 one still can determine α by measuring two successive positive (or negative) peak values x_n and x_{n+2} (Fig. 2.5). It may be shown (see problem 2.8) that

$$\frac{x_n}{x_{n+2}} = e^{\frac{2\pi\alpha}{\omega_p}} \tag{2.15}$$

2.6 Driven Harmonic Oscillator

An important type of motion results when a damped harmonic oscillator is subjected to sinusoidally varying force of the form $F_o \cos \omega t$ where F_o and ω are constants. If such a force is applied to a damped oscillator it is observed after sufficient time has elapsed, that the particle is executing a repetitive type motion which has exactly the same frequency ω as that of the driving force. The equation of motion for such an oscillator is

$$m\ddot{x} + R\dot{x} + Kx = F_o \cos \omega t \tag{2.16}$$

The general solution of this equation consists of the sum of two parts: the general solution of the homogeneous part, $m\ddot{x} + R\dot{x} + Kx = 0$, and any particular solution of the entire equation. The solution of the homogeneous part is exactly that of the damped oscillator which was found in the previous section. The experimental observations suggest that a particular solution might be of the form

$$x = C \sin(\omega t - \theta) \tag{2.17}$$

where C and θ are constants. Differentiating this expression to obtain $\dot{x}$ and $\ddot{x}$ and substituting for x, $\dot{x}$, and $\ddot{x}$ in 2.16 one obtains

$$-C\omega^2 m \sin(\omega t - \theta) + RC\omega \cos(\omega t - \theta) + KC \sin(\omega t - \theta) = F_0 \cos \omega t$$

which on expanding $\sin(\omega t - \theta)$ and $\cos(\omega t - \theta)$ and rearranging becomes

$$\begin{aligned} &\left[C\left\{(\omega^2 m - K)\sin\theta + R\omega\cos\theta\right\} - F_0\right]\cos\omega t \\ &\qquad + C\left[\left[(K - \omega^2 m)\cos\theta + R\omega\sin\theta\right]\sin\omega t = 0\right. \end{aligned} \tag{2.18}$$

This expression must be identically zero, i.e. zero for all possible times if (2.17) is to be a solution of (2.16). It is apparent that if we can make

$$C\left\{(\omega^2 m - K)\sin\theta + R\omega\cos\theta\right\} - F_0 = 0$$

and

$$(K - \omega^2 m)\cos\theta + R\omega\sin\theta = 0$$

by a proper choice of C and Θ, then (2.18) would indeed be identically zero. A choice of Θ such that

$$\begin{tikzcd} & \sin\theta = \frac{\omega m - K/\omega}{\sqrt{R^2 + (\omega m - K/\omega)}} \\ \tan\theta = \frac{\omega m - K/\omega}{R} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & \\ & \cos\theta = \frac{R}{\sqrt{R^2 + (\omega m - K/\omega)^2}} \end{tikzcd} \tag{2.19}$$

will make the second of the above equations correct. One can substitute this value of Θ in the first equation and solve for that value of C which will make the first equation true. One finds

$$C = \frac{F_0/\omega}{\sqrt{R^2 + (\omega m - K/\omega)^2}}$$

A particular solution of (2.16) is thus

$$x = \frac{F_0/\omega \, \sin(\omega t - \theta)}{\sqrt{R^2 + (\omega m - K/\omega)^2}} \quad \text{where} \quad \theta = \tan^{-1}\left(\frac{\omega m - K/\omega}{R}\right) \tag{2.20}$$

and the general solution is

$$x = \underbrace{A e^{-\alpha t} \cos(\omega_b t + \phi)}_{\text{TRANSIENT}} + \underbrace{\frac{(F_0/\omega) \sin(\omega t - \theta)}{\sqrt{R^2 + (\omega m - K/\omega)^2}}}_{\text{STEADY STATE}}$$

where

$$\alpha = \frac{R}{2m} \quad \text{and} \quad \omega_b = \sqrt{K/m - (R/2m)^2}$$

$$= \sqrt{\omega_0^2 - \alpha^2}$$

The first term of the solution is called the transient part since after a sufficient time has elapsed its contribution to x becomes negligibly small. The second term, the particular solution, is called the steady state solution. Note that after the transient part becomes negligible the motion of the particle is simple harmonic with constant amplitude. The system is then said to be in the steady state and its motion is then described by (2.20). For convenience let

$$Z_m = \sqrt{R^2 + (\omega m - K/\omega)^2} \tag{2.21}$$

so that one may write for the steady state

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \theta)$$

$$\dot{x} = \frac{F_0}{Z_m} \cos(\omega t - \theta) \tag{2.22}$$

$$x = -\frac{\quad}{Z_m} \sin(\omega t - \theta)$$

We note that x, $\dot{x}$, $\ddot{x}$, and the driving force $F_0 \cos \omega t$ all vary harmonically with the time, and that all have the same frequency and period, but that in general no two of these quantities are in phase. It should be apparent that x and $\ddot{x}$ differ in phase by 180° and that the driving force $F \cos \omega t$ and x differ in phase by θ. A more complete discussion of the phase relationships will be deferred until a complex solution of (2.16) is developed, since as pointed out earlier, phase relationships are then much more readily apparent.

2.7 Mechanical Resonance

Let us now calculate the rate at which the driving force does work or supplies energy to our driven oscillator in the steady state condition. Recalling that the work done by a force $\vec{F}$ in an infinitesimal displacement $\vec{ds}$ is by definition $dW = \vec{F} \cdot \vec{ds}$, the rate at which work is being done by the force is $\frac{dW}{dt} = \vec{F} \cdot \frac{\vec{ds}}{dt} = \vec{F} \cdot \vec{v}$ where $\vec{v}$ is the velocity. The rate at which the driving force is supplying energy at a given time is thus

$$\frac{dW}{dt} = (F_0 \cos \omega t)\, \dot{x} = \frac{F_0^2}{Z_m} \cos \omega t \cos(\omega t - \theta)$$

The <u>average</u> rate at which this force supplies energy, the average being taken over one cycle, is the work done by this force during one cycle, divided by the time required for one cycle, i.e., divided by the period . We have then

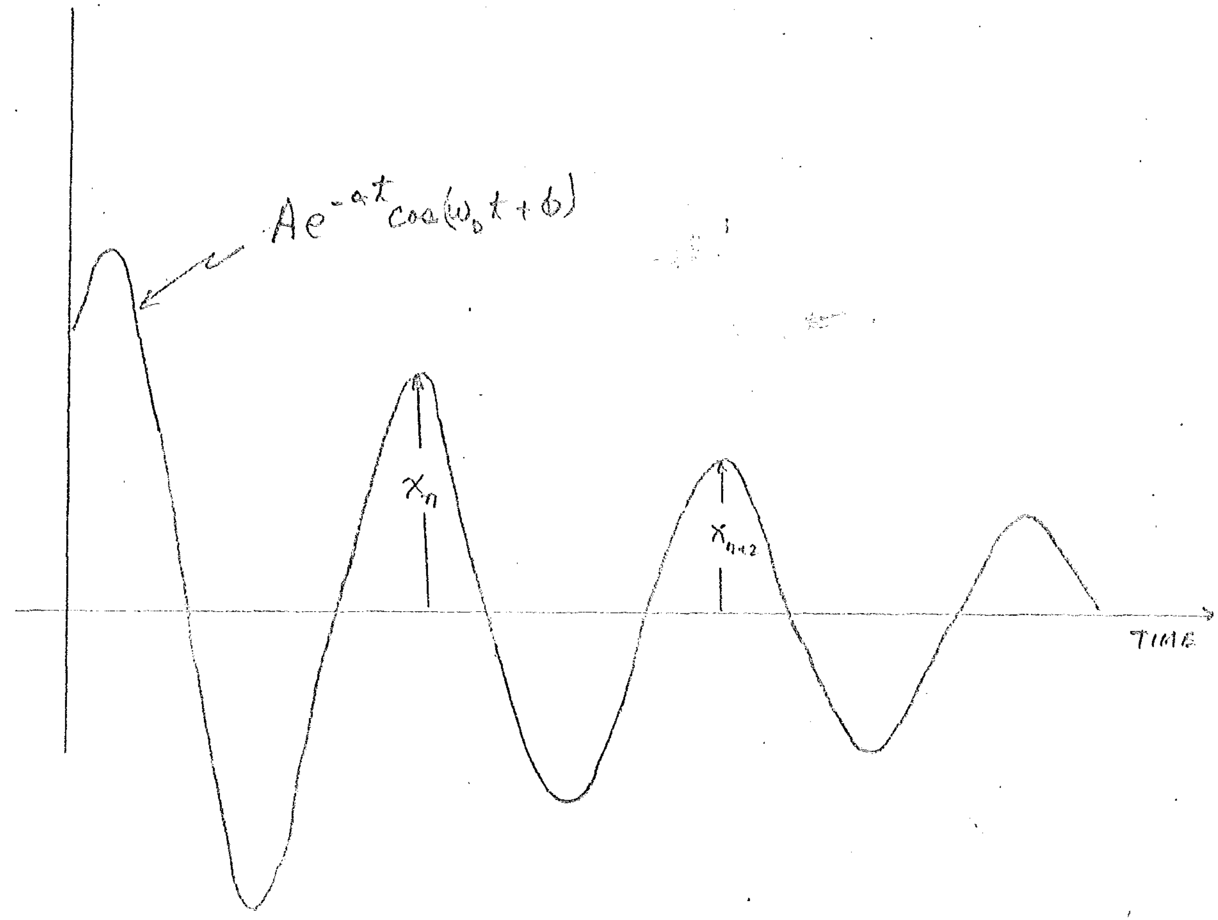

$Ae^{-\alpha t}\cos(\omega_0 t+\phi)$
x_n
x_{n+2}
TIME

Fig 2.5

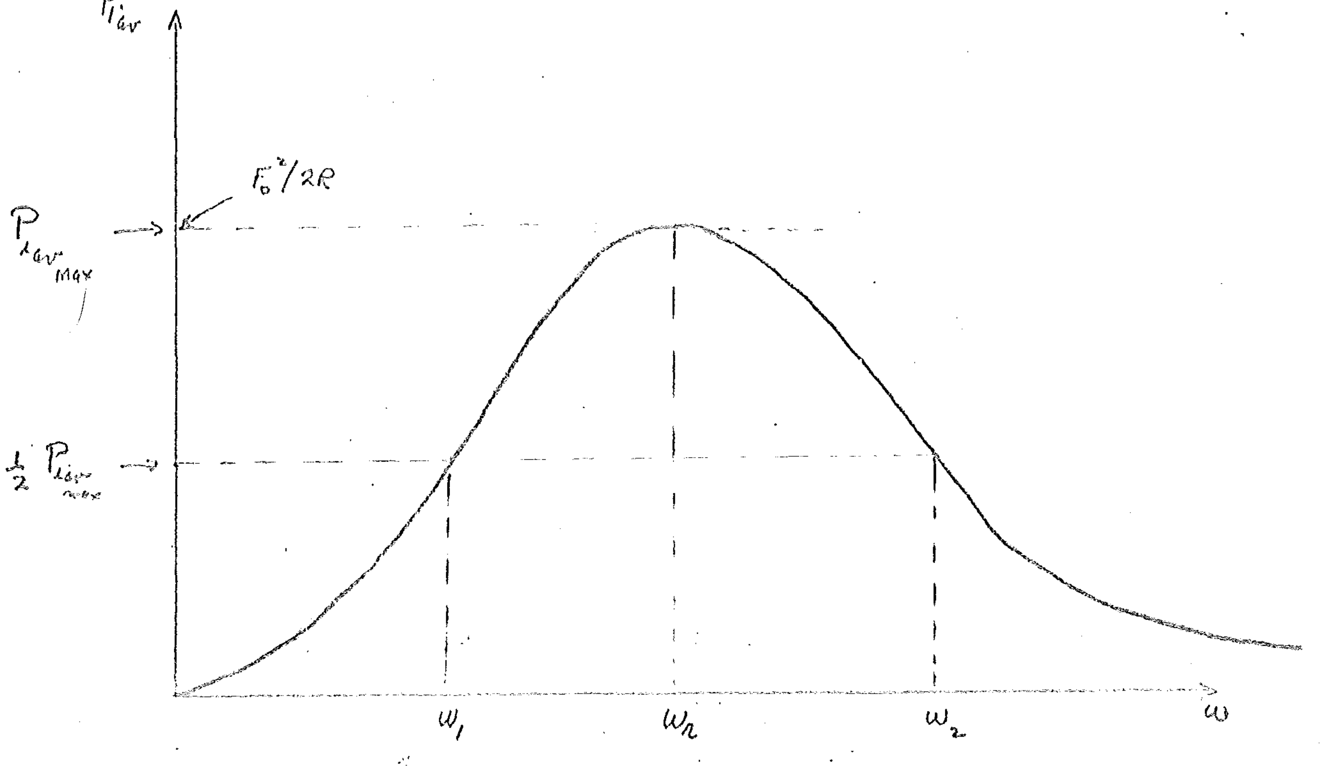

P_{1av}
$F_0^2/2R$
$P_{1av_{max}}$
$\frac{1}{2}P_{1av_{max}}$
ω_1
ω_r
ω_2
ω

Fig 2.6

$$P_{iav} = \left(\frac{dW}{dt}\right)_{av} = \frac{1}{T}\int_0^T \frac{F_o^2}{Z_m}\cos\omega t\cos(\omega t-\theta)\,dt$$

$$= \frac{1}{T}\frac{F_o^2}{Z_m}\int_0^T \cos\omega t\left[\cos\omega t\cos\theta + \sin\omega t\sin\theta\right]dt$$

$$= \frac{F_o^2}{Z_m T}\left\{\frac{\cos\theta}{\omega}\left[\frac{\omega t}{2}+\frac{1}{4}\sin 2\omega t\right]_0^T + \frac{\sin\theta}{\omega}\left[\frac{\sin^2\omega t}{2}\right]_0^T\right\}$$

$$= \frac{F_o^2\cos\theta}{2Z_m}$$

Substituting for $\cos\theta$ from (2.19) and for Z_m from (2.21) this may be written as

$$P_{iav} = \frac{F_o^2 R}{2\left[R^2+(\omega m - K/\omega)^2\right]} \tag{2.23}$$

If the angular frequency ω of the driving force is varied, keeping the amplitude, F_o, of the driving force constant, then P_{iav} will vary since it depends on ω. A plot of P_{iav} as a function of ω, under the condition of constant F_o, is shown in Fig. 2.6. This curve attains a maximum when $\omega = \omega_r = \sqrt{K/m}$ as should be evident from an examination of (2.23). This angular frequency and the corresponding actual frequency at which average input power P_{iav} has its peak value [redacted] the resonant frequency, [redacted] of the system. The actual resonant frequency f_r is given by

$$f_r = \frac{\omega_r}{2\pi} = \frac{1}{2\pi}\sqrt{K/m} \tag{2.24}$$

The resonant frequency and the shape of the P_{iav} versus frequency curve are two important characteristics of an oscillating system. As a quantitative measure of the shape of the curve, one uses a quantity called the Q of the system which is defined by

$$Q = \frac{\omega_r}{\omega_2 - \omega_1} \tag{2.25}$$

where ω_1 and ω_2 are the two angular frequencies at which the input power P_{iav} is 1/2 of the input power at resonance. These two frequencies are indicated in Fig. 2.6. If they lie close to each other then Q is large and P_{iav} decreases rapidly on either side of the resonant frequency, and the resonance is said to be sharp. If ω_1 and ω_2 are widely spaced then Q is small and P_{iav} is approximately constant over a range of frequencies in the neighborhood of the resonant frequency. When this is true the resonance is said to be broad.

One can determine which parameters of the oscillating system determine its Q by calculating ω_1 and ω_2 as follows. If ω' is one of the angular frequencies for which $P_{iav} = 1/2\ P_{iav\,max}$ we have

$$\frac{F_0 R}{2\left[R^2 + (\omega' m - K/\omega')^2\right.} = \frac{1}{2}\left[\frac{F_0}{2R}\right] \tag{2.26}$$

Rearranging and simplifying one obtains

$$\omega' m - K/\omega' = \pm R$$

This equation gives rise to two quadratic equations, one for +R and one for -R. Writing both of these down side by side and solving each for ω' we have:

$$\omega'^2 m + R\omega' - K = 0 \qquad\qquad \omega'^2 m - R\omega' - K = 0$$

$$\omega' = -\frac{R}{2m} \pm \sqrt{(R/2m)^2 + K/\omega} \qquad\qquad \omega' = \frac{R}{2m} \pm \sqrt{(R/2m)^2 + (K/\omega)}$$

There are thus <u>four</u> values of ω' which satisfy (2.26). However, we note that two of these values are negative and have no physical meaning. Setting the larger of the positive values equal to ω_2 and the smaller one to ω_1 yields

$$\omega_2 = \frac{R}{2m} + \sqrt{(R/2m)^2 + K/m}$$

$$\omega_1 = -\frac{R}{2m} + \sqrt{(R/2m)^2 + K/m}$$

Substituting these values in (2.25) gives

$$Q = \frac{\omega_2 m}{R} = \frac{1}{R}\sqrt{Km} \qquad (2.27)$$

Rearranging and simplifying one obtains

$$\omega' m - K/\omega' = \pm R$$

This equation gives rise to two quadratic equations, one for +R and one for -R. Writing both of these down side by side and solving each for ω' we have:

$$\omega'^{2} m + R\omega' - K = 0 \qquad\qquad \omega'^{2} m - R\omega' - K = 0$$

$$\omega' = -\frac{R}{2m} \pm \sqrt{(R/2m)^2 + K/\omega} \qquad\qquad \omega' = \frac{R}{2m} \pm \sqrt{(R/2m)^2 + (K/\omega)}$$

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$$\omega_2 = \frac{R}{2m} + \sqrt{(R/2m)^2 + K/m}$$

$$\omega_1 = -\frac{R}{2m} + \sqrt{(R/2m)^2 + K/m}$$

Substituting these values in (2.25) gives

$$Q = \frac{\omega_2 m}{R} = \frac{1}{R}\sqrt{Km} \tag{2.27}$$

2.8 Complex Form of Solution of the Driven Oscillator

In section 7 we found that the steady state solution of the equation of motion

$$m\ddot{x} + R\dot{x} + Kx = F_0 \cos\omega t \tag{2.28}$$

of a driven harmonic oscillator was

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \theta) \qquad \theta = \tan^{-1} \frac{\omega m - K/\omega}{R}$$

where

$$\dot{x} = \frac{F_0}{Z_m} \cos(\omega t - \theta) \qquad Z_m = \sqrt{R^2 + (\omega m - K/\omega)^2}$$

If one is interested only in the steady state solution as is often the case, it turns out one can obtain such a solution with less algebra by the following technique. Suppose that a force $F_0 \sin\omega t$ rather than $F \cos\omega t$ (this simply means starting to measure time at a different instant) is applied to the oscillator and that y rather than x is used to measure the displacement. The equation of motion in this case would be

$$m\ddot{y} + R\dot{y} + Ky = F_0 \sin\omega t \tag{2.29}$$

If we multiply (2.29) by i and add it to (2.28) we have

$$m(\ddot{x} + i\ddot{y}) + R(\dot{x} + i\dot{y}) + K(x + iy) = F_0(\cos\omega t + i\sin\omega t) \qquad (2.30)$$

which by setting $\underset{\sim}{x} = x + \underset{\sim}{i}y$ can be written

$$m\ddot{\underset{\sim}{x}} + R\dot{\underset{\sim}{x}} + K\underset{\sim}{x} = F_0 e^{i\omega t} \qquad (2.31)$$

If one can find a solution of this complex differential equation of the form

$$\underset{\sim}{z}(t) = x_1(t) + \underset{\sim}{i}\, y_1(t)$$

where $x_1(t)$ and $y_1(t)$ are real functions, it should be apparent that $x_1(t)$ would be a solution of (2.28) and $y_1(t)$ would be a solution of (2.29). Now it is readily verified that the complex function

$$\underset{\sim}{x}(t) = \underset{\sim}{A}\, e^{i\omega t}$$

where

$$\underset{\sim}{A} = -\frac{\underset{\sim}{i}\, F_0/\omega}{R + i(\omega m - K/\omega)}$$

is a solution of (2.31). Hence the real part of

$$\underset{\sim}{x} = -\frac{i(F_0/\omega)\,e^{i\omega t}}{R + i(\omega m - K/\omega)} \qquad (2.32)$$

must be a solution of (2.28). If we write the complex number $R + \underset{\sim}{i}(\omega m = K/\omega)$ in exponential form we have

$$R + i(\omega m - K/\omega) = \sqrt{R^2 + (\omega m - K/\omega)^2}\; e^{i\theta} \quad \text{where} \quad \theta = \tan^{-1}\frac{\omega m - K/\omega}{R}$$

Hence

$$x = -i\,\frac{(F_0/\omega)\,e^{i(\omega t - \theta)}}{\sqrt{R^2 + (\omega m - K/\omega)^2}} = -i\,\frac{(F_0/\omega)}{\sqrt{R^2 + (\omega m - K/\omega)^2}}\Big[\cos(\omega t - \theta) + i\sin(\omega t - \theta)\Big]$$

The real part of this is exactly the steady state solution we found earlier. For reasons mentioned earlier we prefer to regard (2.32) as the steady state solution of the driven harmonic oscillator, and to regard $F_o e^{i\omega t}$ as the driving force. Taking the real part of these complex functions will always give us the actual solution and driving force. If we let

$$\underset{\sim}{Z_m} = R + i(\omega m - K/\omega) \qquad (2.33)$$

we can write

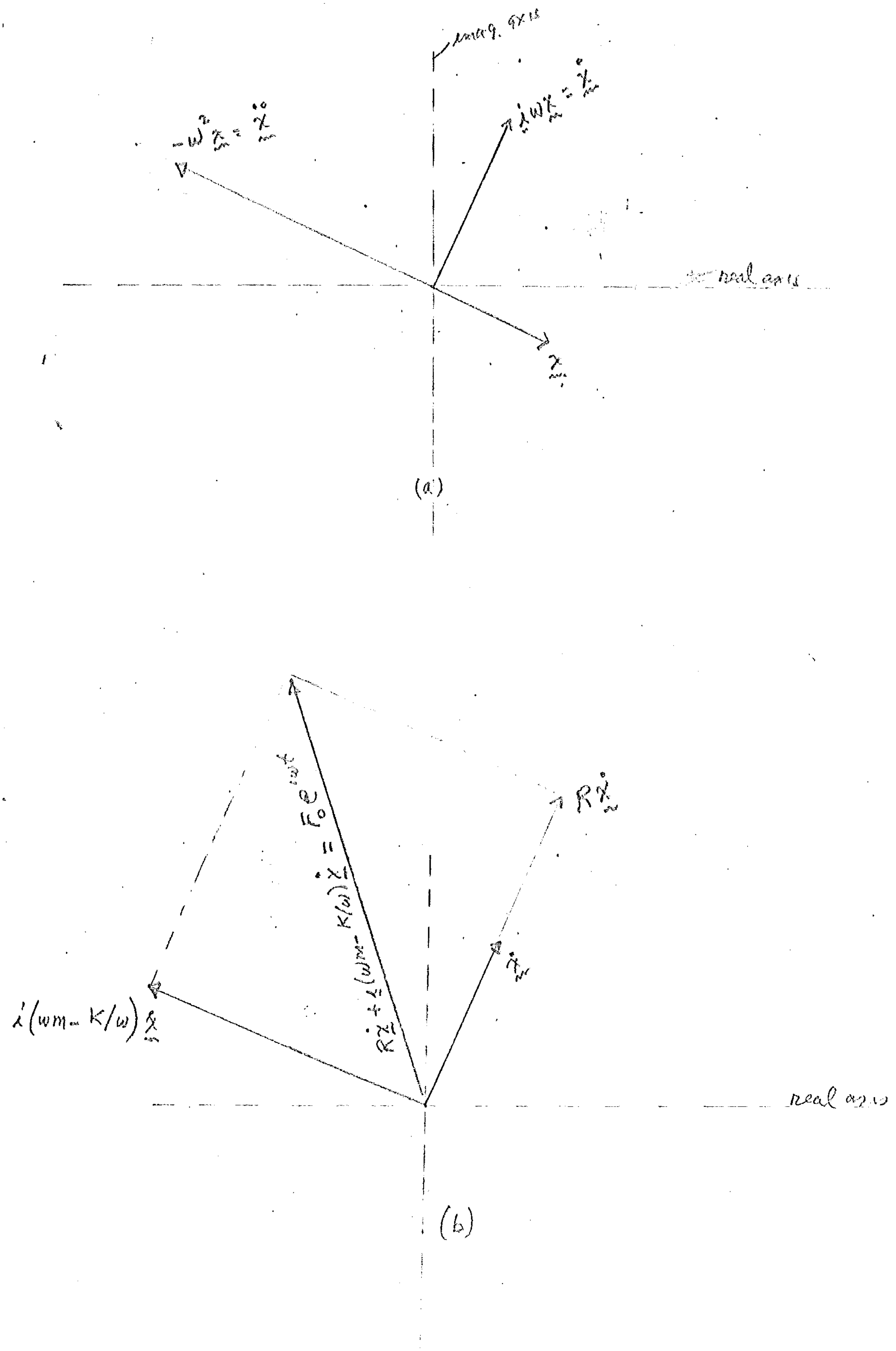
$$\underset{\sim}{x} = -i \frac{F_0/\omega}{\underset{\sim}{Z_m}} e^{i\omega t}$$

$$\dot{\underset{\sim}{x}} = \frac{F_0}{\underset{\sim}{Z_m}} e^{i\omega t} = i\omega \underset{\sim}{x} \tag{2.34}$$

$$\ddot{\underset{\sim}{x}} = i\omega \frac{F_0}{\underset{\sim}{Z_m}} e^{i\omega t} = -\omega_0^2 \underset{\sim}{x}$$

The real parts of $\underset{\sim}{x}$, $\dot{\underset{\sim}{x}}$ and $\ddot{\underset{\sim}{x}}$ correspond exactly to the expression for x, $\dot{x}$ and $\ddot{x}$ given in (2.22). If at an arbitrarily chosen instant of time, one represents $\underset{\sim}{x}$, $\dot{\underset{\sim}{x}}$, $\ddot{\underset{\sim}{x}}$ in the complex plane one obtains a figure like that shown in Fig. 2.7. Although the position of $\underset{\sim}{x}$ is arbitrary since it depends on the particular instant of time chosen, once $\underset{\sim}{x}$ is drawn, the positions of $\dot{\underset{\sim}{x}}$ and $\ddot{\underset{\sim}{x}}$ are fixed from the relation $\dot{\underset{\sim}{x}} = i\omega \underset{\sim}{x}$ and $\ddot{\underset{\sim}{x}} = -\omega_0^2 \underset{\sim}{x}$. Note again that the angle between any two of the quantities is exactly equal to the difference in phase between the corresponding real quantities. Moreover, we note that the (complex) driving force $F_0 e^{i\omega t}$ is related at every instant of time to $\dot{\underset{\sim}{x}}$ by the second of equations (2.34). This may be written

$$F_0 e^{i\omega t} = \underset{\sim}{Z_m} \dot{\underset{\sim}{x}} = \left[R + i(\omega m - k/\omega)\right] \dot{\underset{\sim}{x}} = R\dot{\underset{\sim}{x}} + i\left(\omega m - \frac{k}{\omega}\right) \dot{\underset{\sim}{x}}$$


imag. axis
$i\omega \underset{\sim}{x} = \underset{\sim}{\dot{x}}$
$-\omega^2 \underset{\sim}{x} = \underset{\sim}{\ddot{x}}$
real axis
$\underset{\sim}{x}$
$R\underset{\sim}{\dot{x}}$
$\underset{\sim}{\dot{x}}$
$i(\omega m - K/\omega)\underset{\sim}{\dot{x}}$
$R\underset{\sim}{\dot{x}} + i(\omega m - K/\omega)\underset{\sim}{\dot{x}} = F_0 e^{i\omega t}$
real axis

(a)

(b)

Fig 2.7

If at any instant of time one represents $\underset{\sim}{\dot{x}}$ in the complex plane, then the quantities $R\underset{\sim}{\dot{x}}$ and $\underset{\sim}{i}(\omega m = \frac{K}{\omega})\underset{\sim}{\dot{x}}$ and their sum are fixed as indicated in Fig. 2.7b. From this figure it is easily seen that the angle between the vector representing $F_o e^{i\omega t}$ and that representing $\underset{\sim}{\dot{x}}$ is the angle whose tangent is $(\omega m = K/\omega)/R$ which is the angle Θ defined earlier and is exactly the difference in phase between the driving force $F \cos \omega t$ and the velocity $x = \frac{F_o}{Z_m} \cos(\omega t - \Theta)$. In drawing the figure it was assumed that $\omega m > \frac{K}{\omega}$. For this case the driving force "leads" the velocity by the angle Θ.

Because of the relatively greater ease of manipulation and the fact that the phase relations are more readily apparent, one usually prefers to do algebraic manipulations with the quantities $\underset{\sim}{x}$, $\underset{\sim}{\dot{x}}$, $\underset{\sim}{\ddot{x}}$ and $F_o e^{i\omega t}$ remembering that by taking the real parts of these quantities he can obtain x, $\dot{x}$, $\ddot{x}$ and the real driving force $F_o \cos \omega t$. The technique of working with complex rather than real solutions is almost universally used not only in the study of vibration and sound, but also in the study of electric circuits. It has the rather considerable advantage, not really brought out in the simple examples illustrated, of reducing the solution of a set of differential equations to the solution of a set of algebraic equations involving complex quantities. It should be pointed out that in dealing with energy and power one must use real quantities. In calculating, for example, the average power input as we did in section 7, one must use real values for the force and for the velocity.

$m\ddot{x} + R\ddot{x} + K\dot{x} = F_o \cos \omega t \Rightarrow \frac{F_o e^{i\omega t}}{\underset{\sim}{Z}_m} = \dot{x}(t).$

2.9 Mechanical Impedance

For a driven damped simple harmonic oscillator, the quantities $\underset{\sim}{x}$, $\underset{\sim}{\dot{x}}$, $\underset{\sim}{\ddot{x}}$ and $F_0e^{i\omega t}$ are referred to respectively as the complex acceleration, complex velocity, complex displacement, and complex driving force. The ratio of the complex driving force to the complex velocity is called the mechanical impedance $\underset{\sim}{Z_m}$ of the system. Thus

$$\underset{\sim}{Z_m} = \frac{F_0e^{i\omega t}}{\underset{\sim}{\dot{x}}} = R + i(\omega m - K/\omega)$$

Note that the absolute value of $\underset{\sim}{Z_m}$ is

$$|\underset{\sim}{Z_m}| = Z_m = \sqrt{R^2 + (\omega m - K/\omega)^2}$$

a quantity we had defined earlier. The mechanical impedance $\underset{\sim}{Z_m}$, the driving force $F_0e^{i\omega t}$ and the velocity $\underset{\sim}{\dot{x}}$ play roles in a mechanical system that are analogous to the roles played by the electrical impedance, the applied emf, and the current in an electrical circuit.

2.10 Stiffness, Resistance, and Mass Controlled Oscillators

For a given driven harmonic oscillator it may happen that over a certain range of frequencies one of the three terms R, ωm, or K/ω is much larger than the other two. At frequencies considerably below resonance, for example, K/ω may be much larger than R or ω m. If so then $Z_m \cong K/\omega$ and

$$x \cong \frac{F_0}{K} \sin(\omega t - \theta)$$

Such an oscillator is said to be stiffness controlled over this range of frequencies. Note that it has the important property that the displacement amplitude F_o/K is independent of frequency. Similarly, for frequencies near the resonant frequency of the system, R may be large compared to $(\omega m - K/\omega)$ so that over this range $Z_m \cong R$ and

$$x \approx \frac{F}{\omega R} \sin(\omega t - \theta)$$

$$\dot{x} \approx \frac{F}{R} \cos(\omega t - \theta)$$

Such an oscillator is said to be resistance controlled. Note that although the displacement amplitude is not independent of frequency, the velocity amplitude is. Finally if $\omega m >> \frac{K}{\omega}$ or R then $Z_m \cong \omega m$ and such an oscillator is said to be mass controlled. A mass controlled oscillator has the sometimes desirable property that the acceleration amplitude is independent of frequency.

2.11 The Loudspeaker as a Driven Damped Oscillator

As a practical and sometimes useful example of a system that behaves to a first approximation as a driven damped harmonic oscillator consider the familiar permanent magnet loudspeaker. Two sketches showing the essential features of the loudspeaker are shown in Fig. 2.8. Fastened securely to the center of the speaker cone is a short hollow plastic cylinder on which is wound several turns of copper wire, constituting that is called the voice coil. The speaker cone is flexible allowing some motion of the voice coil along the axis of the cone but subjecting the coil to restoring forces whenever it is moved in either direction from its equilibrium position. The voice coil is positioned so that it lies in a magnetic field set up by a permanent magnet and a soft iron frame. A current I flowing in the voice coil gives rise to a force on the coil, and for a magnetic field $\mathcal{B}$ and a coil length ℓ the force is simply $\mathcal{B} I \ell$ since the field is arranged so that it intersects each element of the coil at right angles. A current $I = I_0 \cos \omega t$ will thus produce a driving force $\mathcal{B} \ell I_0 \cos \omega t$. Motion of the voice coil and speaker cone results in mechanical energy being lost from the system in the form of sound which is radiated and heat which is generated in the cone. In representing the speaker as a driven oscillator we associate these losses with a damping force proportional to the velocity of the voice coil. Thus we write for the equation of motion of the voice coil of the speaker

$$m\ddot{y} + R\dot{y} + Ky = \mathcal{B} \ell I_0 \cos \omega t$$

where y represents the displacement of the voice coil from its equilibrium position. To get better agreement between the predictions of this equation and the actual motion of the voice coil the m should include not only the mass of the voice coil but also some fraction of the speaker cone. The K in the equation depends on the stiffness of the speaker cone. The steady state motion of the voice coil will be given by the real part of

$$\underset{\sim}{\dot{y}} = \frac{B l I_0 e^{i\omega t}}{\underset{\sim}{Z_m}}$$

where

$$\underset{\sim}{Z_m} = R + i\left(\omega m - K/\omega\right)$$

is the mechanical impedance of the speaker.

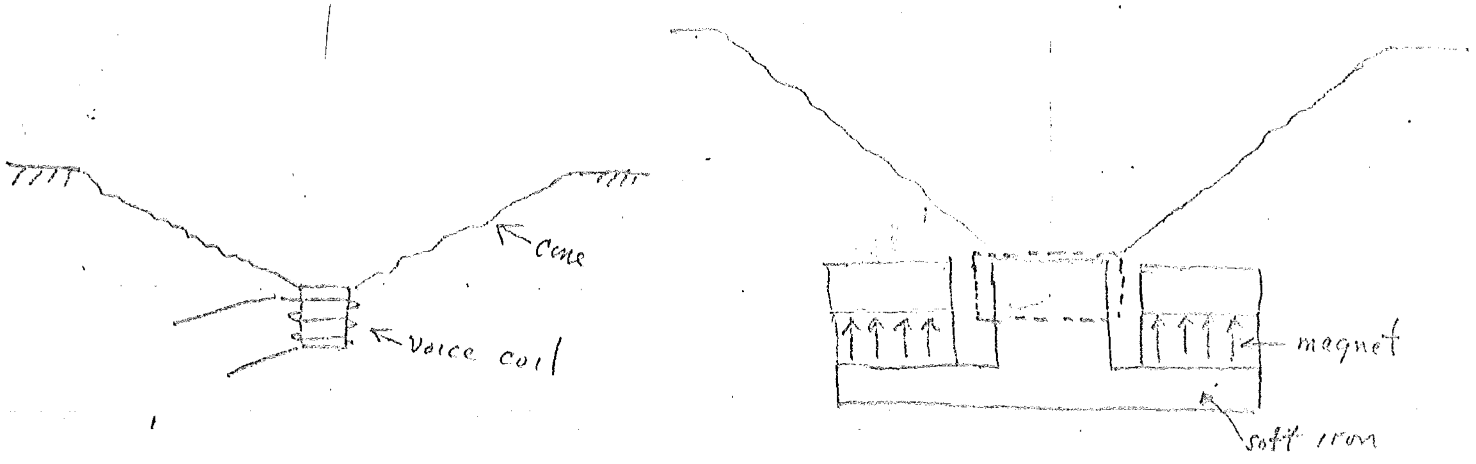


Fig 2.8

Chapter III WAVES ON STRINGS

1. Introduction

Waves and wave motion play an important role not only in the classical areas of acoustics and optics, but also in many areas of modern physics, as the name wave mechanics would suggest. To write down a meaningful definition of a wave is somewhat difficult. However, some concept of what is meant by a wave may be obtained by observing visually the behavior of the system sketched in Fig. 3-1, consisting of a number of blocks of wood fastened at regular intervals to a wire which is suspended from the ceiling. If the lowest block A is given a sudden twist it will be observed that this motion will be transferred to the block immediately above it, causing it to twist, and that the motion will be transmitted in turn to the next block and so on. We describe this motion by saying that a wave is being propagated along the wire. When the motion which is being transferred to successive blocks reaches the block which is fastened to the ceiling, a transfer cannot take place, and one observes that the motion is impressed a second time on the block immediately below the fixed one and subsequently transmitted in turn to each block below it. We say that the wave has been reflected. When the wave reaches the lowest block, a second reflection takes place and the whole process is repeated. Eventually the motion of the block ceases, the initial energy being dissipated in internal friction in the wire.

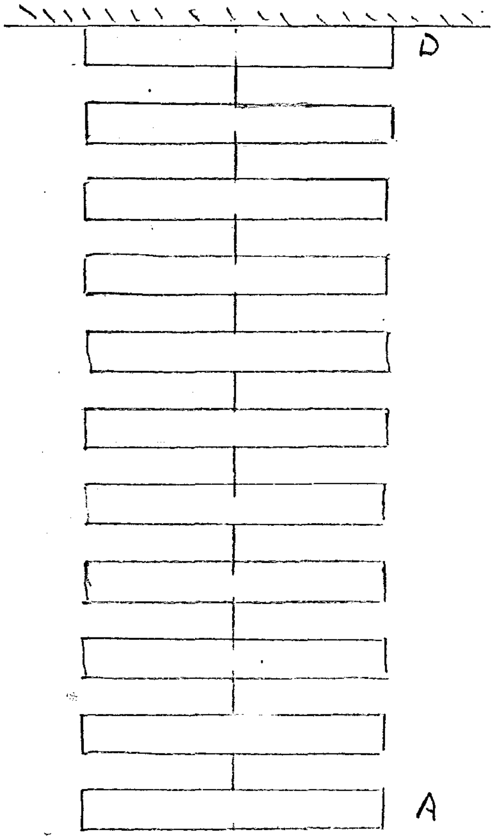


Fig 3,1

In the example above, several characteristics of wave motion may be noted. First there is a definite time required for the motion given to A to be transmitted to any given block above A, i.e., the wave is propagated with a finite velocity. Second, although energy is transferred from block to block along the wire there is no actual transport of mass along the wire. Third, when the wave reaches a point such as D or A where the properties of the medium change, a reflection of the wave takes place.

If block A, instead of being given a sudden twist, is given a periodic motion by twisting it back and forth by hand, one observes after a short time has elapsed that all of the blocks are in motion, oscillating about their equilibrium positions. When this steady state has been established, one no longer can observe that waves are being propagated up and down the wire. All one observes is the regular motion of the individual blocks. Nonetheless, it is reasonable to suppose that waves are still being propagated and that the motions of the individual blocks are produced by these waves.

Although the above system of blocks on a wire is admirably suited for demonstrating waves, it is not the simplest system to analyze mathematically. We consequently will begin by studying transverse waves on a string.

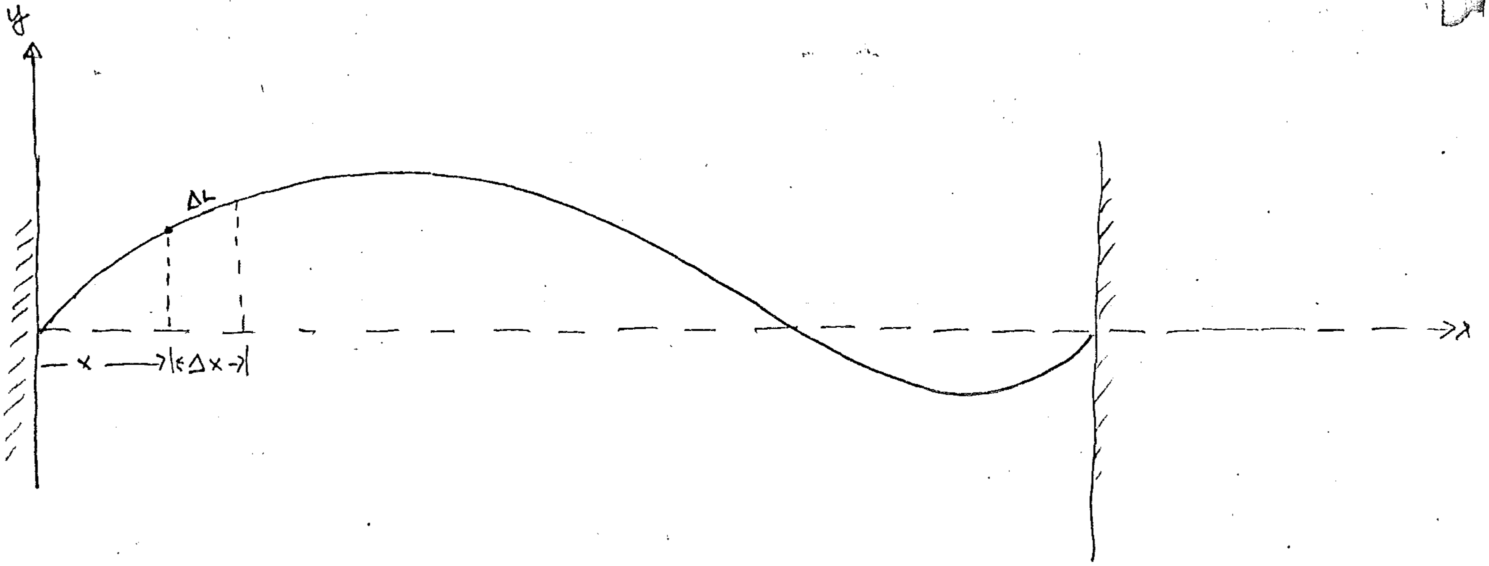


Fig 3.2

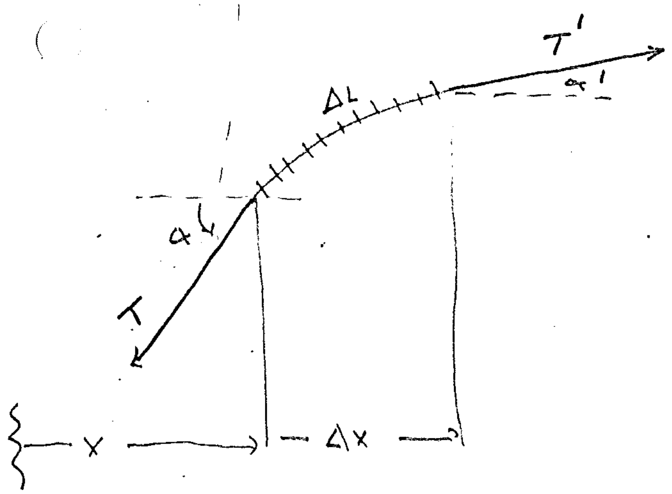


Fig 3.3

2. The wave equation.

It is readily observed that a string fastened between two points and under some tension will vibrate if pulled aside and then released. The wave nature of this motion is not readily apparent; all that we can observe is that each small piece of the string oscillates back and forth in some regular fashion. Nonetheless, as we shall see, the oscillations are readily explained in terms of waves travelling back and forth along the string. First we need to see how one describes the motion of such a string mathematically. Let us assume that the motion is confined to a plane which we will take as the x-y plane. In Fig. 3.2 let the solid line represent the configuration of the string at some instant of time t_1. Using the coordinate system indicated in the figure, we can describe the configuration of the string at time t_1 by some function $y_1(x)$ which if plotted would coincide exactly with the position of the string at every point. At another time t_2, the string would have a different configuration and thus would require a different function $y_2(x)$ to describe it. To completely describe the motion of the string, i.e., to specify its configuration at every instant of time thus requires a large number of functions of x, one for each instant of time. This entire set of functions can be represented formally as $y(x,t)$, each individual function of x being obtained by inserting the corresponding value of time. An equally good way of describing the motion is to specify how each point of the string moves in time. This requires a large number of functions of time, one for each point of the string. This complete set of functions can also be represented by $y(x,t)$, the function of time for a given point being obtained by inserting the x coordinate of that point. Thus the motion of a string vibrating in a plane can always be described by some function $y(x,t)$.

We will now show that <u>any</u> function $y(x,t)$ which describes the motion of a string must meet a certain requirement; it must be a solution of a partial differential equation called the wave equation. This condition comes about by requiring the motion of each small piece of the string be governed by Newton's second law. Referring again to Fig. 3.2 let us isolate for consideration a small piece of string of length ΔL. Fig. 3.3 shows this small piece considerably enlarged and shows the two forces T^1 and T exerted on its two ends by the other portions of the string*. Newton's second law applied to this small piece, assuming it

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* A sketch showing <u>all</u> the forces acting on this piece of string would show in addition a gravitational force and a damping force. For any real string the magnitude of the gravitational force can be shown to be extremely small compared to T and T^1 (see problem 3.1), so that the effect of neglecting it is inconsequential. For real strings, the damping force is <u>not</u> negligible, since it is readily observed that a vibrating string left to itself comes to rest rather quickly. Nevertheless we will neglect the damping forces at this point in our development to keep the mathematics as simple as possible.

moves only in a vertical direction yields the following two equations.

$$T'\cos\alpha' - T\cos\alpha = 0$$

$$T'\operatorname{sen}\alpha' - T\operatorname{sen}\alpha = m a_y$$

Here m is the mass of the piece and a_y stands for the y-component of the acceleration. If the amplitude of vibration of the string at any point is small then the angles α and α' will be small no matter which piece of string or which instant of time we choose. If α and α' are sufficiently small then to a good approximation

$$\cos\alpha = 1 \qquad \cos\alpha' = 1$$

$$\operatorname{sen}\alpha = \tan\alpha \qquad \operatorname{sen}\alpha' = \tan\alpha'$$

If we make these approximations we see that $T = T^1$ and the y equation of motion can be written as

$$T\left[\tan\alpha' - \tan\alpha\right] = m a_y \tag{3.1}$$

If ρ is the mass per unit length of the string, then $\rho\Delta x$ may be written for m. Also if y(x,t) represents the configuration at the instant of time t we are considering

$$\tan\alpha' = \left.\frac{\partial y(x,t)}{\partial x}\right|_{x+\Delta x, t} = f_x(x+\Delta x, t) .$$

$$\tan\alpha = \left.\frac{\partial y(x,t)}{\partial x}\right|_{x, t} = f_x(x, t)$$

Since y(x,t) also specifies how that point of the string a distance x from the end moves in time, the acceleration of the <u>midpoint</u> of the small piece of string under consideration is

$$a_y = \left.\frac{\partial^2 y(x,t)}{\partial t^2}\right|_{x+\frac{\Delta x}{2}, t} = f_{tt}(x+\tfrac{\Delta x}{2}, t)$$

We can now write (3.1) as

$$T\left[f_x(x+\Delta x, t) - f_x(x,t)\right] = \rho \Delta x \, f_{tt}(x+\tfrac{\Delta x}{2}, t)$$

Dividing by Δx and passing to the limit as $\Delta x \to 0$ we have from the definition of a derivative

$$T\left[f_{xx}(x,y)\right] = \rho f_{tt}(x,y)$$

or in <u>slightly different notation</u>

$$\boxed{c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad c = \sqrt{T/\rho}} \qquad (3.2)$$

This is the wave equation for waves on strings. Any function y(x,t) which is to descrive the motion of a string (subject of course to the restrictions and approximations mentioned above) must satisfy this equation. The quantity T in the above equations is called the tension in the string and is equal to the <u>magnitude</u> of the force any given segment of the string exerts on any neighboring segment.

3. Solutions of the wave equation.

Any function y(x,t) which satisfies the partial differential equation (3.2) is said to be a solution of it. Fundamentally, one finds solutions by trial and error, although as we shall see presently, there are general methods of finding solutions which work in many instances. Before looking for any solution we note that (3.2) has the following important property: if one can find two different functions, say $y_1(x,t)$ and $y_2(x,t)$ both of which satisfy (3.2) then their sum or more generally, the function

$$y(x,t) = a\,y_1(x,t) + b\,y_2(x,t) \tag{3.3}$$

where a and b are arbitrary constants, is also a solution.* This so-called superposition property [redacted] may be easily proved as follows. Substituting (3.3) into (3.2) one obtains

$$c^2\left[a\frac{\partial^2 y_1}{\partial x^2} + b\frac{\partial^2 y_2}{\partial x^2}\right] = a\frac{\partial^2 y_1}{\partial t^2} + b\frac{\partial^2 y_2}{\partial t^2}$$

which on rearranging becomes

$$a\left[c^2\frac{\partial^2 y_1}{\partial x^2} - \frac{\partial^2 y_1}{\partial t^2}\right] + b\left[c^2\frac{\partial^2 y_2}{\partial x^2} - \frac{\partial^2 y_1}{\partial t^2}\right] = 0$$

Since y_1 and y_2 both are solutions, the terms in brackets are zero and hence $y = ay_1 + by_2$ is also a solution since it satisfies the differential equation.

---

* Equations which have this property are said to obey the superposition principle.

It is easy to show that any function y(u) where u = x - ct satisfies the wave equation (3.2). We have, using the function of a function rule,

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial y}{\partial u}(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u}\right)\right]\frac{\partial u}{\partial x} = \frac{\partial^2 y}{\partial u^2}(1) \qquad (3.4)$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial y}{\partial u}(-c)$$

$$\frac{\partial^2 y}{\partial t^2} = \left\{\frac{\partial}{\partial u}\left(c\frac{\partial y}{\partial u}\right)\right\}\frac{\partial u}{\partial t} = \frac{\partial^2 y}{\partial u^2}c^2 \qquad (3.5)$$

Substituting y(u) into (3.2) using (3.4) and (3.5) yields an identity proving y(u) is a solution. It should now be evident that any function y(v) where v = x + ct also will satisfy the wave equation, and it should be evident that setting u = ct - x or v = ct + x would not invalidate the argument. By virtue of the superposition property the sum of any function $y_1(u)$ and any other function $y_2(v)$ is also a solution. We assert without proof that this sum, ~~is the general solution of the wave equation in the~~

$$y(x,t) = y_1(x-ct) + y_2(x+ct) \qquad (3.6)$$

is the general solution of the wave equation in the sense that any solution we may find of (3.2) can always be derived from (3.6) by writing some specific function for y_1 or y_2. The function y(x,t) which describes the motion of a vibrating string thus must be of the form (3.6).

Any function y(x-ct) represents a "disturbance" moving to the right with a velocity c. This may be seen from the following

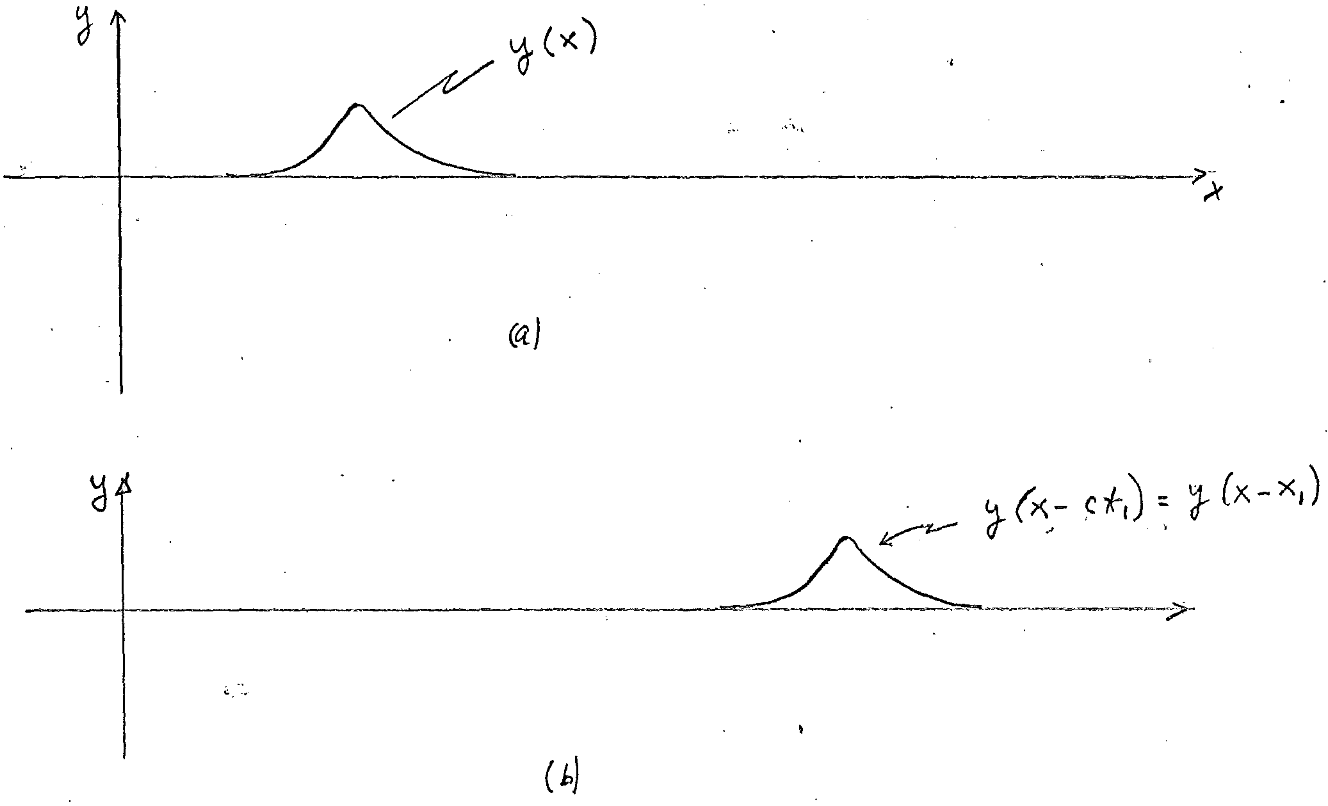


Fig 3.4

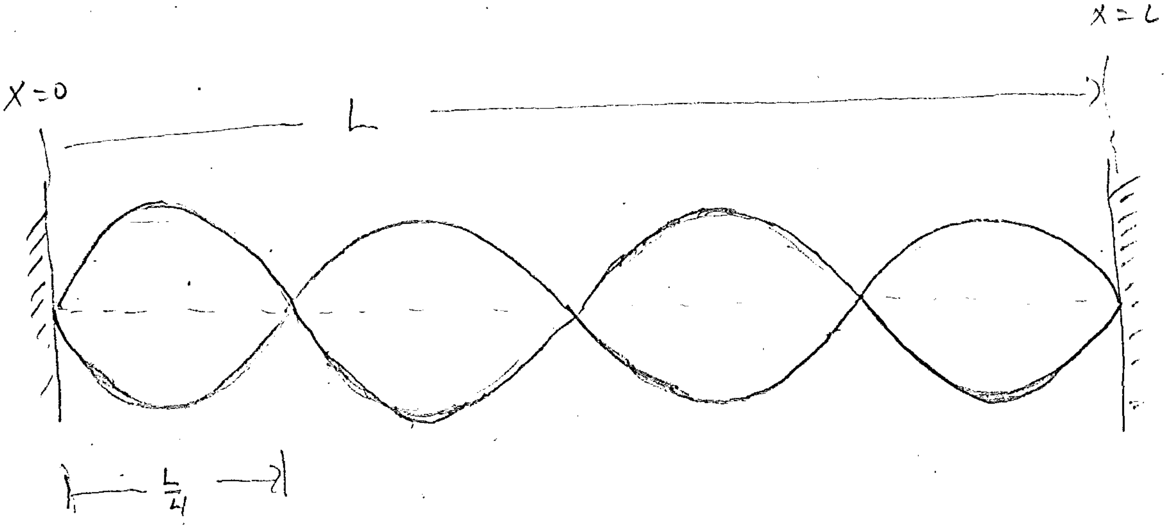


Fig 3.5

considerations. At time $t = 0$, $y(x - ct)$ becomes simply $y(x)$, i.e., some function of x. Suppose, for example, this function when plotted gives the curve shown in Fig. 3.4(a). At another time say t_1, $y(x - ct)$ becomes some other function of x, namely $y(x - ct_1) = y(x - x_1)$ where $x_1 = ct_1$. But we know from analytical geometry that $y(x - x_1)$ has the same form as $y(x)$ except that each point is displaced a distance x_1 to the right. Hence $y(x - ct_1)$ must look as in Fig. 3.4(b). In time t_1 the "disturbance" has moved a distance x_1 to the right, hence must be moving with a speed $c = x_1/_t$. Thus the quantity $c = \sqrt{T/\rho}$ must represent the speed with which a disturbance or wave moves along a string. By a similar argument one can show that any function $y(x + ct)$ represents a disturbance propagating to the left with a speed c.

4. Harmonic solutions of the wave equation.

Although at this point we already know the general solution of the wave equation (3.2), let us imagine this were not the case and we were attempting to find a solution. A very useful technique in finding solutions is to "separate the variables", which in the case of equation (3.2) means to look for solutions of the form

$$y(x,t) = X(x)\, H(t) \tag{3.7}$$

where $X(x)$ is a function of x alone, and $H(t)$ is a function of t only. Substituting (3.7) into (3.2) one obtains after rearranging

$$c^2 \frac{1}{X} \frac{d^2X}{dx^2} = \frac{1}{H} \frac{d^2H}{dt^2} \tag{3.8}$$

If (3.7) is a solution of the wave equation then condition (3.8) must hold, and moreover it must hold at <u>any</u> point of the string for <u>all</u> times, and at <u>any</u> time for <u>all</u> points of the string. Since the left hand side of (3.8), being a function only of x, doesn't change with time, the right hand side of (3.8) must be the same for all times if the two sides are to be always equal. Hence both sides of (3.8) must equal a constant. Calling this constant $-\omega^2$ we obtain from (3.8) the following two <u>ordinary</u> differential equations

$$\left.\begin{aligned} \frac{d^2X}{dx^2} &= -(\omega^2/c^2)\,X \\ \frac{d^2H}{dt^2} &= -\omega^2 H \end{aligned}\right\} \qquad (3.9)$$

If solutions of these ordinary differential equations exist then X(x)H(t) will be a solution of the wave equation. Both of these equations have the same form as the equation of motion of a simple harmonic oscillator. Their general solutions are therefore

$$X(x) = a\cos(\omega/c)x + b\sin(\omega/c)x$$

$$H(t) = d\cos\omega t + e\sin\omega t$$

where a, b, d, and e are arbitrary constants. ~~The~~ A solution of the wave equation thus ~~becomes~~ is

$$y(x,t) = \left[a\cos(\omega/c)x + b\sin(\omega/c)x\right]\left[d\cos\omega t + e\sin\omega t\right]$$

$$= \left[C\cos\tfrac{\omega}{c}x + A\sin\tfrac{\omega}{c}x\right]\cos\omega t + \left[D\cos\tfrac{\omega}{c}x + B\sin\tfrac{\omega}{c}x\right]\sin\omega t \qquad (3.10)$$

Note that this is a solution of the wave equation for every positive value of the constant ω and for completely arbitrary values of the constants A, B, C and D. Note further that if such an equation represented the motion of a string then each point of the string would be moving in simple harmonic motion with an angular frequency ω. For this reason, solutions of the form (3.10) are called harmonic solutions. It is easy to show (see problem 3.3) that the harmonic solution (3.10) can be expressed in terms of functions whose arguments are x - ct and x + ct.

5. Boundary conditions, eigen frequencies.

We have just seen that any function y(x,t) which is to describe the motion of a string must satisfy the wave equation. There is a second restriction. If the string is tied down at both ends as in Fig. 3.1 then obviously the two ends of the string never move. If y(x,t) is to correctly describe the string then

$$y(0,t) = 0$$

$$y(L,t) = 0$$

where L is the length of the string. These, for obvious reasons, are called boundary conditions.

Now (3.10) is a solution of the wave equation. Does it satisfy the boundary conditions? It may be seen by inspection that for x = 0, (3.10) will be zero for all values of t if C and D are taken equal to zero, i.e., the harmonic solution

$$y(x,t) = \sin \frac{\omega}{c} x \left[A \cos \omega t + B \sin \omega t\right] \tag{3.11}$$

does satisfy the first boundary condition. This will also satisfy the second boundary condition if

$$\frac{\omega}{c} L = n\pi \qquad n = 1, 2, 3 \ldots$$

or

$$\omega = n \frac{\pi c}{L} \tag{3.12}$$

Thus harmonic solutions of the wave equation satisfy the boundary conditions <u>only for these special values of ω.</u> These special values of ω and the corresponding actual frequencies, $f = \frac{\omega}{2\pi}$, are referred to as characteristic or eigen frequencies. For <u>each</u> eigen frequency there is a function of the form (3.11) which satisfies both the wave equation and the boundary conditions. These are referred to as characteristic or eigen functions. We list some for reference.

$$y_1(x,t) = \sin \frac{\pi}{L} x \left[A_1 \cos \frac{\pi c}{L} t + B_1 \sin \frac{\pi c}{L} t\right]$$

$$y_2(x,t) = \sin \frac{2\pi}{L} x \left[A_2 \cos \frac{2\pi c}{L} t + B_2 \sin \frac{2\pi c}{L} t\right]$$

$$\vdots$$

$$y_n(x,t) = \sin \frac{n\pi}{L} x \left[A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t\right] \tag{3.13}$$

If the string is vibrating so that the first of these, $y_1(x,t)$, describes its motion, then the string is said to be vibrating in its first or <u>fundamental</u> mode. The corresponding frequency

$f = \omega/2\pi = c/2L$ is called the fundamental frequency. It is the smallest of the allowed frequencies. If the string is vibrating so that its motion is described by (3.13) then it is said to be vibrating in its n<u>th</u> characteristic mode. Note that the frequency f_n corresponding to the n<u>th</u> mode of vibration is n times the fundamental frequency. When the characteristic frequencies of a vibrating system are all integral multiples of the fundamental frequency, they are called harmonics, f_1 being the first harmonic, $f_2 = 2f_1$ the second harmonic, and so on.

Suppose a string is vibrating in its n<u>th</u> characteristic mode. What is the general appearance of the string? Using a little trigonometry, equation (3.13) which describes the n<u>th</u> mode may be written

$$y_n(x,t) = \left[C_n \sin \frac{n\pi}{L} x\right] \cos(\omega_n t + \phi_n) \qquad (3.14)$$

where $\omega_n = \frac{n\pi c}{L}$ and C_n and ϕ_n are constants related to A_n and B_n. If we consider some particular point of the string corresponding to a particular value of x, say x_1, then the quantity in brackets becomes merely a fixed number, the absolute value of which represents the <u>amplitude</u> of the simple harmonic motion of the particular piece of string at that point. This amplitude is, of course, zero at $x = 0$ and $x = L$ and may also be zero at intermediate points; in fact it will be zero for all values of x lying between 0 and L for which

$$\frac{n\pi}{L} x = \pi, 2\pi, 3\pi, \ldots.$$

For example, for the 4th mode, for which $n = 4$, x is zero at points for which

$$x = \frac{L}{4}, \frac{L}{2}, \frac{3}{4}L$$

as well as at 0 and L. Points for which the amplitude of the motion is zero are called nodes. At points midway between the nodes the amplitude of the vibration is a maximum. Such points are referred to as antinodes. Because an object which is vibrating with simple harmonic motion spends much more time near the end points of its motion (the velocity being smaller there) than it does at its midpoint, an object vibrating with a frequency of 30 cps or greater appears to be an observer to be two approximately stationary objects, one at each end point. Thus, a string vibrating in say its fourth characteristic mode appears as shown in Fig. 3.5. Because the pattern *appears* to be stationary it is referred to as a *standing* wave.

6. Initial conditions, general solution.

We have just shown that there are harmonic solutions of the wave equation of the form (3.13) which satisfy the boundary conditions, there being one such solution for each value of ω given by (3.12). It *is* possible for a string to be vibrating so that its motion is described by *one* of these characteristic functions. The cases for which this is true are very special and require that the string be set in motion in a special way. We inquire if it is possible to find a solution which will describe the motion of a string started in an arbitrary way. By virtue of the superposition property ~~principle~~ the *sum* of all the characteristic modes,

$$y(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi}{L} x \left[A_n \cos \frac{n\pi}{L} t + B_n \sin \frac{n\pi}{L} t \right] \qquad (3.15)$$

is itself a solution of the wave equation, and obviously satisfies the boundary conditions. We argue that if the A_n's and B_n's in (3.15) can be chosen so that this sum correctly describes the motion of a string at a given instant of time then it will correctly describe the motion for all subsequent times. Let the given instant of time be t=0 and let the motion of the string at this instant be described by the two functions $y_0(x)$ and $v_0(x)$, the first function specifying the position of each element of the string at t=0 and the second the velocity of each element. If (3.15) correctly describes the string at t=0 we must have*

$$y_0(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x \tag{3.16}$$

$$v_0(x) = \frac{\pi c}{L} \sum_{n=1}^{\infty} n B_n \sin \frac{n\pi}{L} x \tag{3.17}$$

The required values of the A_n's to satisfy (3.16) can be determined by multiplying both sides of (3.16) by $\sin (\frac{m\pi}{L} x)dx$, where m is some integer, and integrating from 0 to L. All of the terms on the right except the term for which m=n will then be found to vanish (see prob. 3.5) yielding

$$\int_0^L y_0(x) \sin \frac{n\pi}{L} x \, dx = A_n \int_0^L \sin^2 \frac{n\pi}{L} x \, dx = A_n \frac{L}{2}$$

---

* The student may recognize the right-hand sides of (3.16) and (3.17) as Fourier series representations of the functions $y_0(x)$ and $v_0(x)$.

or

$$A_n = \frac{2}{L}\int_0^L y_0(x) \sin\frac{n\pi}{L}x \, dx \qquad (3.18)$$

Similarly

$$B_n = \frac{2}{n\pi c}\int_0^L v_0(x) \sin\frac{n\pi}{L}x \, dx \qquad (3.19)$$

As an example consider a string which is released from rest from the position shown in Fig. 3.6. The initial conditions are

$$y_0(x) = \begin{cases} \frac{a}{gL}x & 0 \le x \le gL \\ -\frac{a}{L(1-g)} + \frac{a}{1-g} & gL \le x \le L \end{cases}$$

$$v_0(x) = 0$$

It should be evident that all B_n's are zero. Substituting in (3.18) we have

$$A_n = \frac{2}{L}\int_0^{gL} \frac{ax}{gL} \sin\frac{n\pi}{L}x \, dx + \frac{2}{L}\int_{gL}^{L} \left(-\frac{ax}{L(1-g)} + \frac{a}{(1-g)}\right) \sin\frac{n\pi}{L} \, dx$$

The integrals are readily evaluated using the method of parts yielding

$$A_n = \frac{2a}{(n\pi)^2 g(1-g)} \sin n\pi g \qquad n = 1, 2, 3 \cdots$$

For $a = 1\ cm,\ g \doteq \frac{1}{3},\ L = 100\ cm$ and $c = 10^4\ cm/sec$

the equation describing the motion of the string becomes

$$y(x,t) = .780 \sin \frac{\pi}{100} x \cos 100\pi t + .195 \sin \frac{\pi}{50} x \cos 200\pi t - .049 \sin \frac{\pi}{25} x \cos 400\pi t + \cdots$$

The coefficient of the terms for n=3, 6, 9, ... are zero. A string vibrating in this manner would be said to have the 3rd, 6th, 9th, etc. harmonics missing.

7. Energy considerations.

Suppose a string is vibrating such that its motion is described by a function y(x,t). The kinetic energy U_k of the string at any instant of time say t_1 is the sum of the kinetic energies of all the elemental lengths, i.e.,

$$U_k = \int_0^L \frac{1}{2} \rho dx \left[\frac{\partial y(x,t)}{\partial t}\right]^2$$

where the derivative $\frac{\partial y}{\partial t}$ is evaluated at the given instant of time t_1 and is, of course, a function of x. At time t_1 the string will have some configuration given by $y(x,t_1)$. The potential energy of the string in this configuration is equal to the work done by the tensile forces as the string is moved from this configuration to some arbitrarily chosen standard configuration. For convenience, we will choose the standard configuration to be the configuration of the string when it is at rest (see Fig. 3.7). Now the potential energy of the string in any given configuration is independent of the way the string got to this configuration. (Recall that for conservative forces the work is independent of the path). In calculating the work

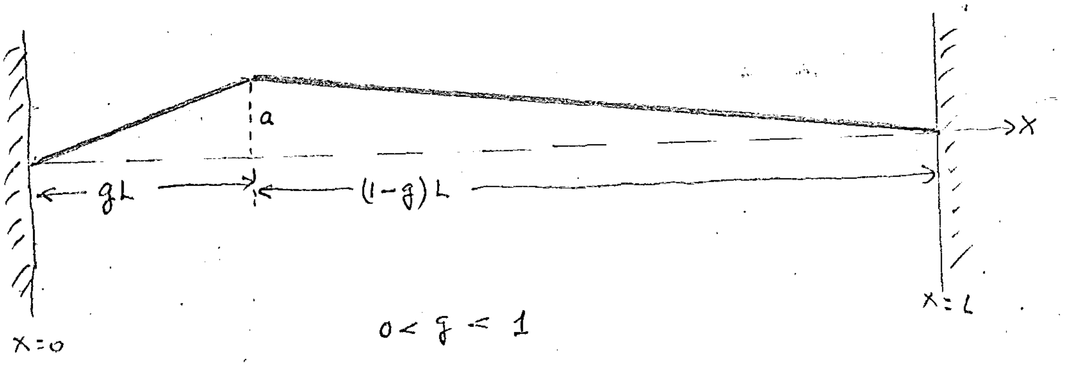
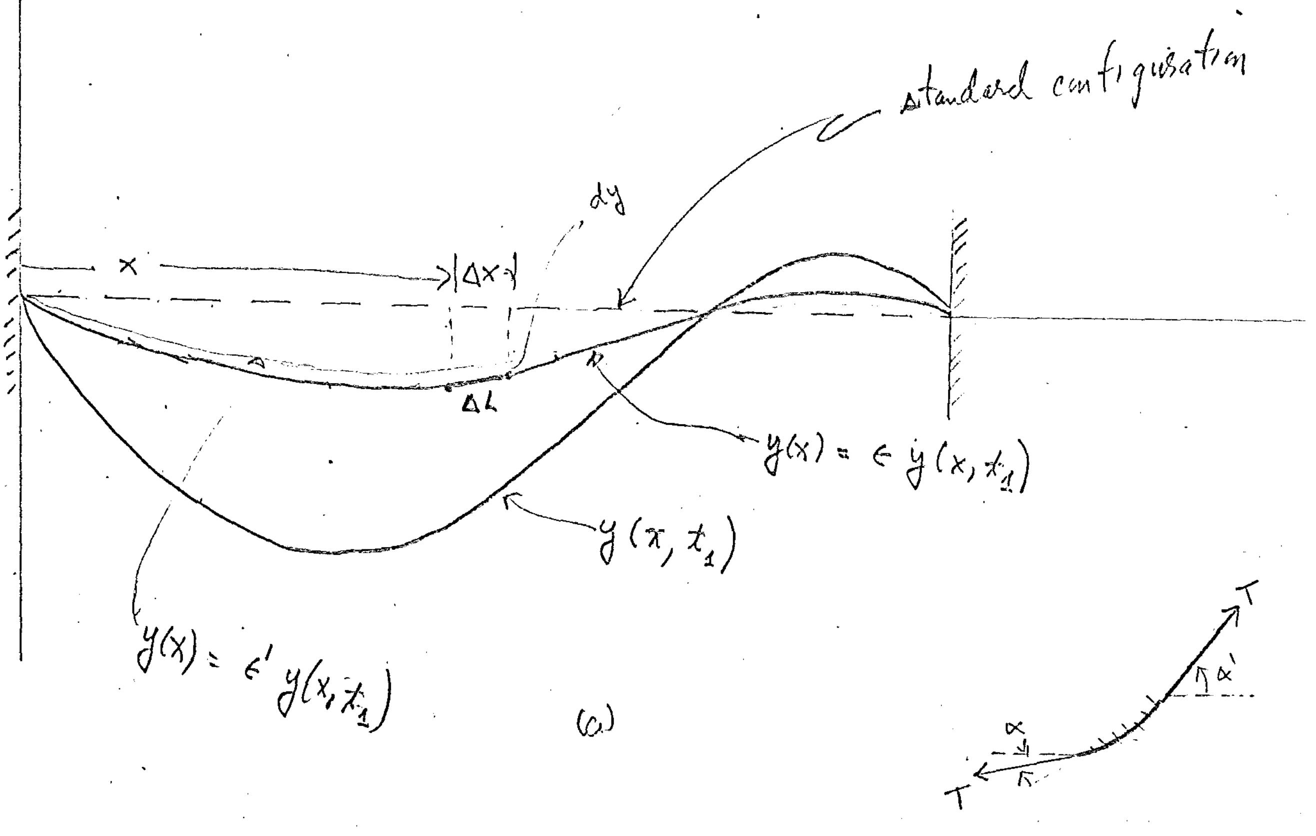


Fig 3.6



(a)

(b)

Fig 3.7

done by the tensile forces we can move the string from the given configuration $y(x,t_1)$ to the standard configuration in any <u>convenient</u> way. We will move the string from the given configuration $y(x,t_1)$ to the standard configuration in such a way that any intermediate configuration between the given and standard will be given by

$$y(x) = \epsilon \cdot y(x,t_1)$$

where ϵ is some positive number between 0 and 1.

Consider the string in one of the intermediate configurations specified by $y(x)$ and isolate a small element of length ΔL. The y-components of the tensile forces acting on the element are

$$T\sin\alpha' - T\sin\alpha \cong T\frac{dy}{dx}\Big|_{x+\Delta x} - T\frac{dy}{dx}\Big|_{x}$$

$$\cong T\left\{ \frac{dy}{dx}\Big|_x + \frac{d}{dy}\left(\frac{dy}{dx}\right)\Delta x + \cdots \right] - T\frac{dy}{dx}\Big|_x$$

$$\cong T\frac{d^2y}{dx^2}\,dx$$

The work done by ~~this~~ these forces as the string is moved from the given to the standard configuration is

$$dU_p = \int_{y(x,t_1)}^{0} \left[T \frac{\partial^2 y}{\partial x^2}\, dx \right] dy$$

Remember that the quantity $\frac{d^2y}{dx^2}$ is evaluated at x and is a function of y, the variable of integration. Now

$$y(x) = \epsilon\, y(x,t_1)$$

$$\frac{d^2y}{dx^2} = \epsilon \frac{\partial^2 y(x,t_1)}{\partial x^2}$$

$$dy = y(x,t_1)\, d\epsilon$$

Substituting one gets

$$dU_p = \int_1^0 T\epsilon \frac{\partial^2 y(x,t_1)}{\partial x^2} dx \left[y(x,t_1)\right] d\epsilon = T \frac{\partial^2 y(x,t_1)}{\partial x^2} y(x,t_1)\, dx \int_1^0 \epsilon\, d\epsilon$$

$$= -\frac{T}{2} \frac{\partial^2 y(x,t_1)}{\partial x^2} y(x;t_1)\, dx$$

Dropping the subscript on the t we have for the potential energy of the entire string when it is in a configuration specified by y(x,t)

$$U_p = -\frac{T}{2} \int_0^L y(x,t) \frac{\partial^2 y(x,t)}{\partial x^2}\, dx$$

This integral may be recast in a different form by integrating using the method of parts Setting

$$u = y \qquad dv = \frac{\partial^2 y}{\partial x^2}\, dx$$

$$du = dy \qquad v = \frac{\partial y}{\partial x}$$

we get

$$\mathcal{U}_p = -\frac{T}{2}\, y\, \frac{\partial y}{\partial x}\Big|_0^L + \frac{T}{2}\int_0^L \frac{\partial y}{\partial x}\, dy$$

$$= 0 + \frac{T}{2}\int_0^L \frac{\partial y}{\partial x}\left(\frac{\partial y}{\partial x}\, dx\right) = \frac{T}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \qquad (3.20)$$

The total energy thus becomes

$$\mathcal{U} = \mathcal{U}_k + \mathcal{U}_p = \frac{\rho}{2}\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{T}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \qquad (3.21)$$

If a string is vibrating in one of its characteristic modes so that its motion is described by

$$y_n(x,t) = \sin\frac{n\pi}{L}x\left[A_n \cos\frac{n\pi c}{L}t + B_n \sin\frac{n\pi c}{L}t\right]$$

$$= C_n \sin\frac{n\pi}{L}x\, \cos\left(\frac{n\pi c}{L}t + \phi_n\right)$$

then

$$\left(\frac{\partial y}{\partial t}\right)^2 = \left(\frac{n\pi c}{L}\right)^2 C_n^2 \sin^2\frac{n\pi}{L}x\, \cos^2\left(\frac{n\pi c}{L}t + \phi_n\right)$$

$$\left(\frac{\partial y}{\partial x}\right)^2 = \left(\frac{n\pi}{L}\right)^2 C_n^2 \cos^2\frac{n\pi}{L}x\, \sin^2\left(\frac{n\pi c}{L}t + \phi_n\right)$$

and (3.20 yields

$$\mathcal{U}_n = \frac{\rho\,(n\pi c)^2}{4L}\, C_n^2 \qquad (3.21)$$

Chapter V. WAVES IN MEMBRANES

If one blows across the top of a thin sheet of plastic (e.g. Saran Wrap) stretched across a rectangular or circular form as in Fig. 5.1 one will hear a characteristic tone. This tome is produced by the vibration of the plastic sheet. It can be inferred by inspection that the amplitude of vibration is very small, since it is difficult to observe with the unaided eye. In developing a description of the motion of such a "membrane" one assumes that the motion of any small piece is strictly at right angles to the plane formed by the undisturbed membrane. If one takes this latter plane as the xy plane, then the motion of the membrane can be described by some function z(x,y,t). Just as in the case of the string it turns out that any function describing the motion must satisfy a wave equation, this condition coming about by the requirement that the motion of any small piece of the membrane must be governed by Newton's second law.

5.1 Wave Equation

Consider first a membrane stretched over a rectangular form of length a and width b. Let the origin of the coordinate system be at one corner of the membrane as indicated in Fig. 5.2. We assume our membrane is homogeneous and isotropic and that the forces applied at the boundaries are uniformly distributed over the perimeter of the membrane as suggested in Fig. 5.3a. With such a uniform distribution, the magnitude of the force on any piece of the perimeter of length ΔL can be expressed as $T \Delta L$ where T is the force per unit length (the sum of the magnitudes of all the forces shown divided by the perimeter). If one isolates for consideration the triangular (shaded) portion of the membrane shown in Fig. 5.3 a and b and asks what forces the adjacent portion of the membrane must exert on this isolated piece, in order that the isolated piece be in equilibrium, one sees that these forces must have a resultant $\vec{R}$ whose x and y components must be numerically equal to $T\Delta L'$ and $T\Delta L$ respectively. This resultant must have a magnitude given by

$$R = \sqrt{(T\Delta L)^2 + (T\Delta L')^2} = T\sqrt{(\Delta L)^2 + (\Delta L')^2} = T\{\text{length of side } \Delta F\}$$

Moreover, it should be evident from geometry that $\vec{R}$ is at right angles to the side ΔF. By extending this argument to other portions of the membrane one arrives at the conclusion that the force that any piece of the membrane exerts on an adjacent portion across the line separating the two is always in the nature of a pull at right angles to the line and has a magnitude equal to T multiplied by the length of the line. The quantity T which is determined by the externally applied forces is called the tension in the membrane.

It follows from the above argument that with the membrane at rest, the forces exerted <u>on</u> a small piece $\Delta x\,\Delta y$ of the membrane by the adjacent portions are as indicated in Fig. 5.4a. In Fig. 5.4b the membrane is shown at some instant of time t after it has been set in vibration. The two forces labelled $T''\Delta y$ and $T'\Delta y$ no longer lie in the xy plane; each makes a small angle with the x-axis, as indicated in Fig. 5.4c which shows the curve formed by the intersection of the membrane with a plane parallel to the xy plane and passing through the center of $\Delta x\,\Delta y$. Since the motion of $\Delta x\,\Delta y$ is assumed to be <u>only</u> in the z-direction, the x-components of $T''\Delta y$ and $T'\Delta y$ must add up to zero. If the angles α'' and α' which these two forces make with the x-axis are sufficiently small so that the cosines may be taken as unity, then

$$T'' \Delta y - T' \Delta y = 0$$

or

$$T'' = T' = T$$

where the last result follows from consideration of an element of area whose edge coincides with one of the boundaries. Since the y-components of the forces on $\Delta x \, \Delta y$ must also add up to zero, it follows that $T_2 = T_1 = T$. Thus the magnitudes of the four forces shown in Fig. 5.4a remain unchanged when the membrane is set in motion; only their direction changes.

The z-components of the two forces $T''\Delta x$ and $T' \Delta x$ is from Fig. 5.4c

$$T\Delta x \sin \alpha'' - T\Delta x \sin \alpha' \cong T\Delta x [\tan \alpha'' - \tan \alpha']$$
$$\cong T\Delta x \left[\frac{\partial z}{\partial x}\Big|_{x+\Delta x, y, t} - \frac{\partial z}{\partial x}\Big|_{x,y,t}\right]$$

Similarly, by considering the curve formed by the intersection of the membrane with a plane parallel to the yz axis and passing through the center of $\Delta x \, \Delta y$, one finds the z-components of the forces $T_2 \Delta x$ and $T_1 \Delta y$ to be

$$T\Delta y \left\{ \frac{\partial z(x,y,t)}{\partial y}\Big|_{x,y,\Delta y,t} - \frac{\partial z(x,y,t)}{\partial y}\Big|_{x,y,t} \right\}$$

Newton's equation of motion for the element thus becomes

$$T\Delta y \left\{ \frac{\partial z}{\partial y}\Big|_{x+\Delta x, y, t} - \frac{\partial z}{\partial y}\Big|_{x,y,t} \right\} + T\Delta y \left\{ \frac{\partial z}{\partial x}\Big|_{x, y+\Delta y, t} - \frac{\partial z}{\partial x}\Big|_{x,y,t} = (\sigma \Delta x \Delta y) \frac{\partial^2 z}{\partial t^2}\Big|_{x+\frac{\Delta x}{2}, y+\frac{\Delta y}{2}, t}\right.$$

where σ is the mass per unit area of the membrane. Dividing through by $\Delta x \, \Delta y$ and passing to the limit one obtains

$$T\frac{\partial^2 z}{\partial x^2} + T\frac{\partial^2 z}{\partial y^2} = \sigma\frac{\partial^2 z}{\partial t^2}$$

or

$$\boxed{c^2\left[\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right]=\frac{\partial^2 z}{\partial t^2} \quad ; \quad c=\sqrt{T/\sigma}} \qquad (5.1)$$

This is the wave equation for waves in membranes and any function z(x,y,t) which is to describe the motion of a [redacted] membrane must be a solution of this wave equation.

It is a simple matter to demonstrate that any function f(u) where

$$u = ct - (x \cos \theta + y \sin \theta)$$

is a solution of the wave equation (5.1) for arbitrary values of θ. That functions f(ct - [x cos θ + y sin θ]) have wave properties can easily be seen by choosing a new coordinate system X, Y where axes are inclined at an angle θ to the xy axis as indicated in Fig. 5.5. For any point P, the x and y coordinates are related to the X and Y coordinates by

$$X = x \cos \theta + y \sin \theta \quad ; \quad Y = y \cos \theta - x \sin \theta$$

Hence f(ct - [x cos θ + y sin θ]) becomes f(ct - X). ~~Thus~~ This we recognize as a disturbance being propagated in the +X direction with a velocity c. Hence any further f(ct - [x cos θ + y sin θ]) represents a disturbance being propagated in a direction making an angle θ to the positive x axis.

5.2 Harmonic Solutions, Boundary Conditions, Eigen Functions

The general approach for finding solutions of partial different equations is to separate the variables, i.e. to look for solutions of the form

$$z(x,y,t) = X(x)Y(y)H(t) \tag{5.2}$$

where X(x) is a function of x, ~~and~~ Y(y) is a function of y only and H(t) is a function of t only. Substituing (5.2) into the wave equation one obtains after rearranging the following expression

$$c^2 \left[\frac{1}{x} \frac{d^2X}{dx^2} + \frac{1}{Y} \frac{d^2Y}{dy^2} \right] = \frac{1}{H} \frac{d^2H}{dt^2}$$

If (5.2) is a solution, the above expression must hold for all values of x,y and t. Since the left-hand side is only a function of x and y it doesn't change with t, and hence the right-hand side must be the same for all times, i.e. equal to a constant. Calling this constant $-\omega^2$ we obtain the following two ordinary differential equations

$$\frac{1}{H} \frac{d^2H}{dt^2} = -\omega^2$$

$$\frac{1}{X} \frac{d^2X}{dx^2} = -\left(\frac{\omega}{c}\right)^2 - \frac{1}{Y} \frac{d^2Y}{dy^2}$$

The general solution of the first of these should be immediately apparent. It is

$$H(t) = C_3 \cos \omega t + D_3 \sin \omega t$$

where C_3 and D_3 are arbitrary constants.

The second equation must hold for all x and y if (5.2) is to be a solution. Again this leads to the conclusion that both sides must be equal to a constant. Calling this constant $-\alpha^2$ we obtain the following two differential equations

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\left[\left(\frac{\omega}{c}\right)^2 - \alpha^2\right]$$

We can write down the general solutions of these two equations immediately since they are of the same form as (5.2) provided $\frac{\omega}{c} > \alpha$. We obtain

$$X = C_1 \cos \alpha x + D_1 \sin \alpha x$$

$$Y = C_2 \cos \sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\, y + D_2 \sin \sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\, y$$

Our solution of the form (5.2) is thus

$$z(x,y,t) = \left[C_1 \cos \alpha x + D_1 \sin \alpha x\right]\left[C_2 \cos\sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\, y + D_2 \cdot \sin\sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}\, y\right]\left[C_3 \cos \omega t + D_3 \sin \omega t\right] \tag{5.3}$$

This is a solution for every value of ω and every value of and for arbitrary values of the constants C_1, C_2, C_3, D_1, D_2, D_3. If such a function did describe the motion of the membrane, then any point (x,y) of the membrane would be moving in simple harmonic motion with a frequency ω. For this reason (5.3) is called an harmonic solution.

If the membrane is stretched over a rectangular form of dimensions a and b, then function, z(x,y,t), describing the motion of the membrane must satisfy the following boundary conditions:

(i) $z(0,y,t) = 0$

(ii) $z(a,y,t) = 0$

(iii) $z(x,0,t) = 0$

(iv) $z(x,b,t) = 0$

If we examine the harmonic solution (5.3) it is apparent that if we choose C_1 and C_2 both equal to zero, conditions (i) and (iii) will be satisfied. Moreover, we can satisfy condition (ii) for arbitrary values of D_1 if we restrict γ to values given by

$$\alpha = \frac{m\pi}{a} \qquad m = 1,2,3, \ldots\ldots\ldots$$

and we can satisfy condition (iv) for arbitrary values of D_2 if we restrict $\sqrt{\left(\frac{\omega}{c}\right)^2 - \alpha^2}$ to values given by

$$\sqrt{\left(\frac{\omega}{c}^{2}\right) - \alpha^2} = \frac{n\pi}{b} \qquad n = 1,2,3,\ldots\ldots\ldots$$

We see from these two restrictions that the harmonic solution (5.3) will satisfy the boundary conditions only for values of ω given by

$$\omega = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad \begin{matrix} m = 1,2,3 \ldots\ldots\ldots \\ n = 1,2,3 \ldots\ldots\ldots \end{matrix}$$

and hence for frequencies

$$f = \frac{\omega}{2\pi} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

These values of ω and f are of course the eigen frequencies and the corresponding functions,

$$z_{mn}(x,y,t) = \sin\frac{m\pi}{a}x \, \sin\frac{n\pi}{b}y \left[A_{mn}\cos\left(c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, t\right) + B_{mn}\sin\left(c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, t\right)\right]$$

are the eigen functions of a membrane with a rectangular boundary. There is an eigen function for any combination of values of m and n. The smallest of the eigen frequencies

$$\omega_{11} = c\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$$

is called the fundamental frequency and if the membrane is vibrating so that its motion is described by the corresponding eigen function

$$z_{11}(x,y,t) = \sin\frac{\pi}{a}x \sin\frac{\pi}{b}y\left[A_{11}\cos\left(c\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\,t\right) + B_{11}\sin\left(c\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\,t\right)\right] = \left[C_{11}\sin\frac{\pi}{a}x \sin\frac{\pi}{b}y\right]\cos(\omega_{11}t+\phi_{11})$$

it is said to be vibrating in its fundamental mode. If it is vibrating in its fundamental mode, the amplitude of the motion (represented by the quantity in the brackets) is a maximum at the center of the membrane, since the two sine terms in the bracket have a value of one at that point. Since $\sin\frac{\pi}{a}x \sin\frac{\pi}{b}x$ is positive for <u>every point</u> of the membrane, if at any time $z_{11}(x,y,t)$ is positive for any one point it will be positive for every other point; the motion of any point of the membrane is thus in phase with the motion of every other point.

If a membrane is vibrating so that it is described by the eigen function for which m=2 and n=3 i.e. the function

$$z_{23} = \sin\frac{2\pi}{a}x \sin\frac{3\pi}{b}y\left[A_{23}\cos\omega_{23}t + B_{23}\cos_{23}t\right]$$

$$= \left[C_{23}\sin\frac{2\pi}{a}x \sin\frac{3\pi}{b}y\right]\cos(\omega_{23}t+\phi_{23})$$

$$\omega_{23} = c\sqrt{\left(\frac{3\pi}{a}\right)^2+\left(\frac{3\pi}{b}\right)^2}$$

then it should be apparent that the amplitude will be zero for any point for which

$$x = \frac{a}{2}$$

and zero for any point for which

$$y = \frac{b}{3}\,,\ \frac{2b}{3}$$

Hence, in addition to the boundaries there will be nodal lines as indicated by the dotted lines in Fig. 5.6. Note that the quantity $\sin\frac{2\pi}{a}x\ \sin\frac{3\pi}{b}y$ is positive for every point in the shaded regions of Fig. 5.6 and negative for every point in the unshaded regions. If then at some instant of time $z_{23}(x,y,z,t)$ is positive for one of the points in the shaded regions it will be positive for every point in the shaded regions and negative for every point in the unshaded regions. Thus the motions of any two points in the shaded region are in phase, and are 180° out of phase with the motion of any point in the unshaded region.

5.4 General Solution.

The sum

$$z(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{b}\left[A_{mn}\cos W_{mn}t + B_{mn}\cdot \sin W_{mn}t\right]$$

$$W_{mn} = \sqrt{\left(\frac{m\pi c}{a}\right)^2 + \left(\frac{n\pi c}{b}\right)^2}$$

of all the eigen functions is itself a solution of the wave equations satisfying the boundary conditions. It may be regarded as a general solution in the sense that with the proper choice of the A_{mn}'s and the B_{mn}'s it will describe the motion of a membrane started in vibration in an arbitrary way (subjected, of course, to the limits on the amplitude for which our approximations are reasonably valid). If one knows the z coordinate and the velocity of every point of the membrane at some instant of time, say t=0, then one can determine the A_{mn}'s and the B_{mn}'s such that (5.4) will describe its subsequent motion. If

$$z_0(x,y)$$
$$v_0(x,y)$$

are the functions describing the position and velocity of each point of the membrane at t = 0, then

$$z_0(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \, A_{mn}$$

Multiplying both sides by $\sin \frac{m'x}{a} \sin \frac{n'y}{b} dxdy$ and integrating over the surface of the membrane one obtains

$$\int_0^b \int_0^a z_0(x,y) \sin \frac{m'x}{a} \sin \frac{n'y}{b} \, dx\, dy =$$
$$\int_0^b \int_0^a \sin \frac{m'\pi}{a} x \sin \frac{n'\pi}{b} y \, dx dy \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$$

Although the double sum on the right looks more formidable than the single sum we obtained in the case of strings, if one writes out a few terms of this double sum, it will be seen that the integration is perfectly straight forward, all integrals being zero except those for m = m' and n = n'. For m = m' and n = n' the integration on the right yields $\frac{ab}{4}$ so that

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b z_0(x,y) \sin \frac{mx}{a} \sin \frac{ny}{b} dxdy$$

Similarly one obtains

$$B_{mn} = \frac{4}{ab\omega_{mn}} \int_0^a \int_0^b v_0(x,y) \sin \frac{mx}{a} \sin \frac{ny}{b} dxdy$$

5.5 Circular Boundary, Wave Equation in Polar Coordinates

For a membrane with a circular boundary, Fig. 5.7 a and b, the external forces are presumed to be distributed uniformly around the boundary so that the magnitude of the force exerted on any small segment of length ΔL of the boundary can be written as $T\Delta L$, where T is a constant called the tension. By requiring that each portion of the membrane be in equilibrium, one can show by an argument similar to that used in section 5.1 that the force that any portion of the membrane exerts on an adjacent portion across the line separating the two is always in the nature of a pull at right angles to the line and has a magnitude equal to T multiplied by the length of the line. If the motion of each piece of the membrane is perpendicular to the plane of the undisturbed membrane, the motion can be described by some function $z(r,\phi,t)$.

Fig. 5.7d shows the forces exerted on a small segment of the membrane of area $r\Delta\phi\,\Delta r$, when the membrane is at rest. Fig. 5.7e shows at some instant of time t after the membrane has been set in motion, the curve formed by the intersection of the membrane with the radial plane $z = \phi + \frac{\Delta\phi}{2}$. The two forces labelled $T(r+\Delta r)\Delta\phi$ and $T'r\Delta\phi$ in Fig. 5.7d are labelled $T''(r+\Delta r)\Delta\phi$ and $T'r\Delta\phi$ in Fig. 5.7e. Writing down Newton's second law for the r-motion one has at this instant of time

$$T''(r+\Delta r)\Delta\phi\cos\psi'' - T'r\Delta\phi\cos\psi' = \sigma r\Delta r\Delta\phi\ a_r$$

where a_r is the radial component of acceleration of the midpoint of the segment. If the angles ψ'' and ψ' are at every instant sufficiently small, then since there is no radial motion, $a_r = 0$ and one obtains on dividing by $\Delta\phi$ and passing to the limit as Δr goes to zero

$$T'' - T' = 0$$

or

$$T'' = T' = T$$

where the last result follows by considering a small segment whose outer edge coincides with the boundary of the membrane. The z-components of the two forces $T''(r + \Delta r)\ \phi$ and $T' r \Delta\phi$ can now be written

$$T(r+ \Delta r)\,\Delta\phi \left.\frac{\partial z(r,\phi,t)}{\partial r}\right|_{r+ \Delta r,\phi,t} - T\ r\Delta\phi \left.\frac{\partial z}{\partial r}\right|_{r,\phi,t}$$

where we have used the approximation that $\sin \gamma'' \cong \tan \gamma'' = \left.\frac{\partial z}{\partial r}\right|_{r + \Delta r}$ and $\sin \gamma' = \tan \gamma' = \left.\frac{\partial z}{\partial r}\right|_{r}$. In a similar manner, by considering the curve formed by the intersection of the membrane with the cylinder $z = r + \frac{\Delta r}{2}$, one can show that the vertical components of the two forces labelled $T\Delta r$ in Fig. 5.7d are at time t given by

$$T \Delta r \left.\frac{\partial z(r,\phi,t)}{r\partial \phi}\right|_{r,\phi+ \Delta\phi,t} - T \Delta r \left.\frac{\partial z(r,\phi,t)}{r.\Delta\phi}\right|_{r,\phi,t}$$

Newton's second law for the z-motion of the element $r\Delta r\Delta\phi$ becomes

$$T(R+ \Delta r)\,\Delta\phi \left.\frac{\partial z}{\partial r}\right|_{r+ \Delta r,\phi,t} - Tr\Delta\phi \left.\frac{\partial z}{\partial r}\right|_{r,\phi,t}$$

$$+ T \Delta r \left.\frac{\partial z}{r\partial\phi}\right|_{r,\phi+ \Delta\phi,t} - T \Delta r \left.\frac{\partial z}{r\partial\phi}\right|_{r,\phi,t} = \sigma\ r\ \Delta r\,\Delta\phi \left.\frac{\partial^2 z}{\partial t^2}\right|_{r+ \frac{\Delta r}{2},\ \phi+ \frac{\Delta\phi}{2},\ t}$$

Dividing by $r\,\Delta r\,\Delta\phi$ and passing to the limit as both Δr and $\Delta\phi$ go to zero one obtains

$$T\left[\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\,\frac{\partial z}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 z}{\partial \phi^2}\right] = \sigma\,\frac{\partial^2 z}{\partial t^2}$$

or

$$\boxed{c^2\left[\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\,\frac{\partial z}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 z}{\partial \phi^2}\right] = \frac{\partial^2 z}{\partial t^2} \qquad c = \sqrt{T/\sigma}} \tag{5.5}$$

This is the wave equation expressed in polar coordinates.

5.6 Harmonic Solution, Bessel Functions

If there are solutions of the wave equation of the form

$$Z(r,\phi,t) = R(r)\,\Phi(\phi)\,H(t) \tag{5.6}$$

then substitution into (5.5) leads to the condition

$$c^2\left[\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\,\frac{dR}{dr}\right) + \frac{1}{\Phi r^2}\,\frac{d^2\Phi}{d\phi^2}\right] = \frac{1}{H}\,\frac{d^2 H}{dt^2}$$

which must hold for all times and for all values of r and ϕ. It follows that both sides must equal the same constant. Calling this constant $-\omega^2$ leads to the following two equations

$$\frac{d^2 H}{dt^2} = -\omega^2 H \tag{5.7}$$

$$r^2\left[\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\,\frac{dR}{dr}\right) + \frac{\omega^2}{c^2}\right] = -\frac{1}{\Phi}\,\frac{d^2\Phi}{d\phi^2} \tag{5.8}$$

Since the latter of these equations must hold for all values of ϕ and all values of r, each side must equal the same constant. Calling this constant m^2 leads to the following two differential equations

$$\frac{d^2 \Phi}{d\emptyset^2} = - m^2 \Phi$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2}\right) R = 0 \qquad (5.10)$$

where $k = \omega/c$. If one can find solutions of (5.7), (5.9) and (5.10) then there exists a solution of the form $R(r)\,\Phi(\emptyset)H(t)$. Solutions of (5.7) and (5.9) are readily apparent:

$$H(t) = A \cos \omega t + B \sin \omega t$$

$$\Phi(\emptyset) = A' \cos m\emptyset + B' \cos m\emptyset$$

Assuming one can find some function say R(r) which satisfies (5.10) one will have an harmonic solution of the form

$$z(r,\emptyset,t) = R(r) \left[A'\cos m\emptyset + B'\sin m\emptyset\right] \left[A \cos \omega t + B \sin \omega t)\right] \qquad (5.11)$$

If this function is actually describing the motion of a membrane then the motion of a point located say at r_1, $\emptyset_1$ is given by $z(r_1,\emptyset_1,t)$. Since the point located at $(r_1,\emptyset_1)$ and the one at $(r_1,\emptyset_1 \pm \ell 2\pi)$ where ℓ is any integer are <u>exactly the same point</u> of the membrane it follows that for the description of the motion to be unambiguous $z(r_1,\emptyset_1,t) = z(r_1,\emptyset_1 \pm \ell 2\pi,t)$. Equation (5.11) will have this required property only if the constant m is restricted to integral values, i.e.

$$m = 0,\ 1,\ 2,\ 3\ .....$$

Keeping in mind that m must have integral values, we attempt to find a solution of (5.10) by assuming one exists of the form

$$R(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots = \sum_{n=0}^{\infty} a_n r^n \qquad (5.12)$$

where a_1, a_2 ... are constants. It follows that

$$\frac{1}{r}\frac{dR}{dr} = a_1r^{-1} + 2a_2 + 3a_3r + 4a_4r^2 + 5a_5r^3 + \ldots\ldots$$

$$\frac{d^2R}{dr^2} = 2a_2 + 6a_3r + 12a_4r^2 + 20a_5r^3 + \ldots\ldots,$$

$$k^2R = k^2a_o + k^2a_1r + k^2a_2r^2 + k^2a_3r^3 + \ldots\ldots$$

$$\frac{-m^2}{r^2}R = -m^2a_or^{-2} -m^2a_1r^{-1} -m^2a_2 -m^2a_3r -m^2a_4r^2 -m^2a_5r^3 + \ldots$$

Substituting into (5.10) one gets

$$\frac{-m^2a_o}{r^2} + \frac{(1-m^2)a_1}{r} + \left[(4-m^2)a_2 + k^2a_o\right]r^o + \left[(9-m^2)a_3 + k^2a_1\right]r \qquad (5.13)$$

$$+ \left[(16-m^2)a_4 + k^2a_2\right]r^2 + \left[(25-m^2)a_5 + k^2a_3\right]r^3 \ldots = 0$$

Remembering that this expression must be zero for <u>all</u> values of r if (5.12) is to be a solution, it is apparent that <u>either</u> m <u>or</u> a_o must be zero, and <u>either</u> $(1-m^2)$ <u>or</u> a_1 must be zero, since otherwise the first and second terms become infinite at r = 0. If m = 0, setting a_1, a_3, a_5..... equal to zero and choosing

$$a_2 = -\frac{k^2}{4}a_o$$

$$a_4 = -\frac{k^2}{16}a_2 = \frac{k^4}{(16)(4)}a_o$$

$$a_6 = -\frac{k^2}{36}a_4 = -\frac{k^6}{(36)(16)4}a_o$$

will make (5.13) identically zero for any arbitrary choice of a_o.

For $m = 1$ setting a_o, a_2, a_4, a_6, ... all equal to zero and choosing

$$a_3 = -\frac{k^2}{8} a_1$$

$$a_5 = -\frac{k^2}{24} a_3 = \frac{k^4}{(24)(8)} a_1$$

$$a_7 = -\frac{k^2}{48} a_5 = \frac{k^6}{(48)(24)8} a_1$$

will make (5.13) identically zero for any arbitrary choice of a_1.

For $m = 2$ setting a_o and a_1, a_3, a_5, a_7 .... all equal to zero and choosing

$$a_4 = -\frac{k^2}{12} a_2$$

$$a_6 = -\frac{k^2}{32} a_4 = \frac{k^4}{(32)(12)} a_2$$

$$a_8 = -\frac{k^2}{60} a_4 = -\frac{k^6}{(60)(32)(12)} a_2$$

will make (5.13) identically zero for an arbitrary choice of a_2. Thus the following are solutions of (5.10):

$$m = 0, \quad R(r) = a_o\left[1 - \frac{(kr)^2}{4} + \frac{(kr)^4}{(16)(4)} - \frac{(kr)^6}{(36)(16)(4)} + \ldots\right]$$

$$= a_o\left[1 - \frac{(\frac{kr}{2})^2}{1!1!} + \frac{(\frac{kr}{2})^4}{2!\ 2!} - \frac{(\frac{kr}{2})^6}{3!\ 3!} + \ldots\right]$$

$$= a_o\left[J_o(kr)\right]$$

$$m = 1, \quad R(r) = a_1\left[r - \frac{k^2r^3}{8} + \frac{k^4r^5}{(24)(8)} - \frac{k^6r^7}{(48)(24)(8)} + \ldots\right]$$

$$= \frac{2a_1}{k}\left[\frac{(\frac{kr}{2})}{0!1!} - \frac{(\frac{kr}{2})^3}{1!\ 2!} + \frac{(\frac{kr}{2})^5}{2!3!} - \frac{(\frac{kr}{2})^7}{3!4!} + \ldots\right]$$

$$= \frac{2a_1}{k}\left[J_1(kr)\right]$$

$$m = 2, \quad R(r) = a_2 \left[r^2 - \frac{k^2}{12} r^4 + \frac{k^4}{(32)(12)} r^6 - \frac{k^6 r^8}{(60)(32)12} + \dots \right]$$

$$= \frac{8a_2}{k} \left[\frac{(kr/2)^2}{0!2!} - \frac{(kr/2)^4}{1!\,3!} + \frac{(kr/2)^6}{2!\,4!} - \frac{(kr/2)^8}{3!\,5!} + \dots \right]$$

$$= \frac{8a_2}{k} \left[J_2(kr) \right]$$

and so on. As indicated above, $J_0(kr)$, $J_1(kr)$ and $J_2(kr)$ are shorthand notations for the infinite series contained in the brackets of the above solutions. The infinite series for which $J_0(kr)$ stands is called the zero order Bessel function of the first kind. Similarly $J_1(kr)$ and $J_2(kr)$ are referred to respectively as the first and second order Bessel functions of the first kind. A plot of these functions (Fig. 5.8) shows that each of these functions resembles a decaying sine function. Some of the more interesting and useful properties of these functions are summarized in Table 5.1.

It should now be evident that there exist for every ~~indegral~~ integral value of m an harmonic solution of the wave equation of the form.

$$z_m(r,\phi,t) = J_m(kr) \left[A'_m \sin m\phi + B'_m \cos m\phi \right] \left[A_m \cos \omega t + B_m \cos \omega t \right]$$

$$= C_m J_m(kr) \left[\sin(m\phi + \alpha_m) \right] \left[\cos(\omega t + \Omega_m) \right] \qquad (5.14)$$

Each of these harmonic solutions is a solution for every positive value of k and for arbitrary values of A'_m, B'_m, A_m and B_m (or C_m, α_m and Ω_m).

5.7 Eigen Frequencies, Eigen Functions, Characteristic Modes for Circular Membrane

If the radius of the circular membrane is a ~~than~~ then the boundary condition is that

$$z_m(a, \emptyset, t) = 0$$

An examination of (5.14) that this will be satisfied if $J_m(ka) = 0$. Every Bessel function of ~~order m~~ the first kind is zero for certain values of the argument. These values determine the eigen frequencies. For example

$$J_0(ka) = 0 \quad \text{for} \quad ka = 2.405,\ 5.520,\ 8.654,\ \ldots$$

$$J_1(ka) = 0 \quad \text{for} \quad ka = 3.832,\ 7.016, 10.174,\ \ldots$$

$$J_2(ka) = 0 \quad \text{for} \quad ka = 5.136,\ 8.417, 11.620,\ \ldots$$

Since k = ω/c, the eigen frequencies for m = 0, 1 and 2 are

$$\begin{array}{ccc}
m = 0 & m = 1 & m = 2 \\
\omega_{01} = \dfrac{2.405}{a} c & \omega_{11} = \dfrac{3.832}{a} c & \omega_{21} = \dfrac{5.136}{a} c \\
\omega_{02} = \dfrac{5.520}{a} c & \omega_{12} = \dfrac{7.016}{a} c & \omega_{22} = \dfrac{8.417}{a} c \\
\omega_{03} = \dfrac{8.654}{} c & \omega_{13} = \dfrac{10.174}{a} c & \omega_{23} = \dfrac{11.620}{a} c
\end{array}$$

The corresponding eigen functions are

$$z_{01} = C_{01} J_0\left(\frac{2.405}{a} r\right) \cos\left(\frac{2.405}{a} ct + \Omega_{01}\right)$$

$$z_{02} = C_{02} J_0\left(\frac{5.520}{a} r\right) \cos\left(\frac{5.520}{a} ct + \Omega_{02}\right)$$

$$z_{11} = C_{11} J_1\left(\frac{3.832}{a} c\right) \sin(\emptyset + \alpha_{11}) \cos\left(\frac{3.832}{a} ct + \Omega_{11}\right)$$

$$z_{12} = C_{12} J_1\left(\frac{7.016}{a} c\right) \sin(\emptyset + \alpha_{12}) \cos\left(\frac{7.016}{a} ct + \Omega_{12}\right)$$

$$z_{21} = C_{21} J_2\left(\frac{8.654}{a} c\right) \sin(2\emptyset + \alpha_{21}) \cos\left(\frac{8.654}{a} ct + \Omega_{21}\right)$$

The smallest of the eigen frequencies is ω_{01} and the corresponding actual frequency $f_{01} = \omega_{01}/2\pi$ is called the <u>fundamental</u> frequency. If the membrane is vibrating so that its motion is described by z_{01} it is said to be vibrating in its <u>fundamental mode</u>. Since z_{01} is not a function of ϕ, the fundamental mode exhibits circular symmetry. A plot of $J_0\left(\frac{2.405}{a}r\right)$ as a function of r is shown in Fig. 5.9.a. Since this is everywhere positive, it follows that all points of the membrane vibrate in phase, and the membrane vibrates as suggested in Fig. 5.9 b and c.

If the membrane is vibrating so that its motion is described by z_{02} then it should be evident from Fig. 5.9.c. that the motion of all points of the membrane for which $r > 2.405\ a/5.520$ is 180° out of phase with the motion of those points for which $r < 2.405a/5.520$. The motion of the membrane is as indicated in Figs. 5.9d and e.

The modes for which $m \neq 0$ are slightly more difficult to describe, since the amplitude at any point depends on ϕ as well as r. For the mode described by z_{11}, a plot of $J\left(\frac{3.832}{a}r\right)$ as a function of r, Fig. 5.10a, reveals that this function is positive for $r < a$. However, a plot of $\cos(\phi + \alpha_{11})$ as a function of ϕ shows it is positive for $\phi < \frac{\pi}{2} - \alpha_{11}$ and negative for $\frac{\pi}{2} - \alpha_{11} < \phi < \frac{3}{2}\pi - \alpha_{11}$. There is a nodal line, $\phi = \frac{\pi}{2} - \alpha_{11}$, and the motion of points on one side of this line is 180° out of phase with the motion of points on the other side as suggested in Fig. 5.10c. Figures 5.10 d, e and f suggest how the motion of the mode described by z_{12} may be deduced. This mode exhibits two nodal lines and one nodal circle. Table 5.2 lists the nodal patterns for the modes corresponding to the ten smallest eigen frequencies.

5.8 The Kettledrum

A kettledrum consists of a membrane stretched over the open end of a hemispherical vessel as suggested in Fig. 5.11. When the membrane is at rest, the air trapped in the vessel will be at atmospheric pressure, the same as the air outside, so that the net force on any small area of the membrane due to the pressure of the air is zero. If the membrane is depressed slightly, the volume of the trapped air will decrease and the pressure will increase. The increase in pressure will give rise to a net force on each element of area ΔS of the membrane, the magnitude of the net force being $(P - P_0)\Delta S$ where P is the pressure of the trapped air and P_0 is the pressure of the air outside. If the depression in the membrane is small, the direction of the net force on any element of area will make a very small angle with the vertical, so that the vertical component of the net force is to a good approximation equal numerically to the magnitude of the force.

If the membrane instead of being depressed statically, is set into vibration, the pressure of the air in the vessel will vary above and below atmospheric.* Let us assume that the air in

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* Strictly speaking one can only refer to the pressure of a gas when the gas is in equilibrium, and the pressure is the same at all points. Any sudden motion of the membrane sets up a pressure wave in the air and the air attains equilibrium only after this wave is sufficiently attenuated. In treating the kettledrum, one generally assumes that at each instant the pressure of the trapped air is the pressure the air would attain if the membrane were held fixed in its position long enough for equilibrium to be established. This is a reasonable assumption if the pressure wave is attenuated in a time that is short compared to the period of vibration of the membrane.

the vessel behaves as an ideal gas and that the time for one pressure cycle is short compared to the time for appreciable heat transfer to take place between the trapped air and its surroundings, i.e. assume that the compressions and expansions of the trapped air take place adiabatically. It follows that at every instant

$$PV^{\gamma} = \text{a constant}$$

where P and V are the pressure and volume of the trapped air at that instant, and γ is the ratio of the specific heat of air at constant pressure to that at constant volume. If the pressure changes are sufficiently small it follows that

$$P - P_o \approx dP = - \frac{P_o}{V_o} dV$$

where V_o is the volume of the trapped air when the membrane is at rest. At any instant when the volume of the trapped air differs by dV from the equilibrium value V_o, the vertical component of the force on any area dA due to the pressure differential will be

$$(P-P_o) \Delta S = - \frac{P_o}{V_o} dV \Delta S$$

If one writes down Newton's second law for the element of area Δ S, and includes this force along with the forces due to the tension one obtains after dividing by ΔS and passing to the limit, the following wave equation

$$T \left[\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \phi^2} \right] - \frac{\gamma p_o}{V_o} dV = \sigma \frac{\partial^2 z}{\partial t^2} \qquad (5.15)$$

Any function describing the motion of the membrane must be a solution of this equation. Suppose the membrane is vibrating so that its motion is described by the function $z(r,\phi,t)$. Then at any t, for an element of area r dr dϕ located at r,ϕ, the quantity z(r,ϕ,t) r dr dϕ is the volume of air in the column of length z and area r dr dϕ, shown in Fig. 5.12. This quantity is positive if $z > 0$ and negative if $z < 0$. Hence at time t, the change dV of the volume of the air in the vessel is

$$dV = \int_0^{2\pi}\int_0^{a} z(r,\phi,t)\ r\ dr\ d\phi$$

If z(r,ϕ,t) is of the form $\psi(r,\phi)H(t)$, then

$$dV = H(t)\int_0^{2\pi}\int_0^{a}\psi(r,\phi) r\ dr\ d\phi = I_0\ H(t)$$

where

$$I_0 = \int_0^{2\pi}\int_0^{a}\psi(r,\phi) r\ dr\ d\phi \tag{5.16}$$

is a constant. Thus, if the motion is being described by a function $\psi(r,\phi)H(t)$, then since it must be a solution of the wave equation one must have

$$T\left[H\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}H\frac{\partial\psi}{\partial r} + \frac{1}{r^2}H\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{\gamma P_0}{V_0}I_0 H = \sigma\psi\frac{d^2H}{dt^2}$$

or

$$\frac{c^2}{\psi}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \frac{\gamma P_0 I_0}{\sigma V_0\psi} = \frac{1}{H}\frac{d^2H}{dt^2}$$

Since the quantity on the left is only a function of r and ϕ and that on the left only a function of t, both quantities must equal the same constant, say $-\omega^2$. Thus

$$\frac{d^2H}{dt^2} = -\omega^2 H$$

and

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = \frac{\gamma P_0 I_0}{\sigma V_0 c^2} \tag{5.17}$$

where $k = \omega/c$. The solution of the first of these equations is apparent. To find a solution to the second suppose for the moment the term, on the right, were zero, so the equation were simply

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2} + k^2\psi = 0$$

Assuming a solution of this latter equation exists of the form $R(r)\,\Phi(\phi)$, one obtains on substituting and rearranging

$$r^2\left[\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} + k^2\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$

But this is exactly equation (5.7) whose solution was found to be

$$J_m(kr)\left[A'_m \cos m\phi + B'_m \sin m\phi\right] \qquad m=0,1,2,3\ \ldots$$

Since this is a solution of (5.17) when the right-hand term is zero, and since the right-hand term is a constant it follows that solution of (5.17) exists of the form

$$\Psi(r,\phi) = J_m(kr)\left[A'_m \cos m\phi + B'_m \sin m\phi\right] + K$$

where

$$K = \frac{P_0 I_0}{V_0 c^2}\,\frac{1}{k^2} = \frac{\gamma P_0 I_0}{\sigma w^2 V_0}$$

Thus solutions of the wave equation (5.14) exist of the form

$$z(r,\phi,t) = \left\{J_m(kr)\left[A' \cos m\phi + B' \cos m\phi\right] + \frac{\gamma P_0 I_0}{\sigma \omega^2 V_0}\right\} \left\{A_m \cos \omega t + B_m \sin \omega t)\right\} \tag{5.18}$$

and for each integral value of m there is a solution for each positive value of ω. From (5.16)

$$I_o = \int_0^{2\pi}\int_0^{a}\left\{J_m(kr)\left[A_m' \cos m\phi + B_m' \cos m\phi\right] + \frac{\gamma P_o I_o}{\sigma \omega^2 V_o}\right\} r\, dr\, d\phi$$

or

$$I_o = \left[\frac{1}{1 - \frac{\gamma P_o \pi a^2}{\sigma \omega^2 V_o}}\right] \int_0^{2\pi}\int_0^{a}\left\{J_m(kr)\left[A_m' \cos m\phi + B_m' \sin m\phi\right]\right\} r\, dr\, d\phi$$

Unless $m \neq 0$, $I_o = 0$ because the integral of $\sin m\phi$ and $\cos m\phi$ from 0 to 2π is zero. If $I_o = 0$ then (5.18) reduces to (5.14), i.e. for $m \neq 0$, the harmonic solutions of the kettledrum are <u>identical</u> with the harmonic solutions of the free membrane. Since the boundary conditions are identical for both the kettledrum and the free membrane, it follows that $m \neq 0$, the eigen frequencies and the eigen functions of the kettledrum and the free membrane are identical.

For $m = 0$

$$I = \frac{2\pi}{1 - \frac{\gamma P_o \pi a^2}{\sigma \omega^2 V_o}} \int_0^{a} J_o(kr)\, r\, dr$$

$$= \frac{2\pi}{1 - \frac{\gamma P_o \pi a^2}{\sigma \omega^2 V_o}} \left[\frac{a^2 J_1(ka)}{ka}\right]$$

where the last result is taken from Table 5.1. Harmonic solutions of (5.15) for $m = 0$ are

$$z_0(r,t) = \left\{ J_0(kr) + \frac{\gamma P_0}{\sigma\omega^2 V_0}\left[\frac{2\pi a^2 J_1(ka)}{\left\{1 - \frac{\gamma P_0 \pi a^2}{\sigma\omega^2 V_0}\right\} ka}\right]\right\}\left(A\cos\omega t + B\sin\omega t\right)$$

The boundary condition requires that

$$J_0(ka) + \frac{\gamma P_0}{\sigma\omega^2 V_0}\left[\frac{2\pi a^2\ J_1(ka)}{\left\{1 - \frac{P_0\ a^2}{{}^2V_0}\ ka\right\}}\right] = 0$$

By using the identity $J_0(ka) + J_2(ka) = 2{}_1J\ (ka)/ka$ the above condition may be written

$$J_0(ka) = -\ \frac{J_2(ka)}{(ka)^2} \qquad (5.19)$$

where

$$= \frac{P_0\ a^4}{c^2V_0} \qquad \frac{P_0\ a^4}{T\ V_0}$$

Finding the values of $k = \omega/c$ which satisfy (5.19) will yield the eigen frequencies of the kettledrum for m = 0. Note that if $\alpha = 0$, these eigen frequencies are identical with those of the free membrane. If $\alpha \ll 1$, then one would expect that the eigen frequencies would differ only slightly from their values when $\alpha = 0$. Note that α is made smaller by increasing the tension or by increasing the volume, as one might suspect since both such increases tend to make the tensile forces larger in relation to the pressure forces. The eigen frequencies determined from (5.19) for several numerical values of α are shown in Table 5.3. Note that the fundamental frequency is the one most affected by $\alpha \neq 0$.

Table 5.3

$\alpha = \frac{\gamma P_o \pi a^4}{T V_o}$	k_1a	k_2a	k_3a
0	2.405	5.520	8.654
2	2.68	5.55	8.66
10	3.485	5.67	8.69

5.9 The Driven Membrane, Circular Boundary

If a loudspeaker is mounted some distance from a free membrane as in Fig. 5.13, and the speaker is driven at some frequency ω determined by the oscillator setting, then the sound wave emitted by the speaker will cause the pressure P on the top surface of the membrane to vary with time in the following manner

$$P = P_0 + P_1 \cos\omega t$$

where P_0 is atmospheric pressure, and P_1 is a constant which depends on how hard the speaker is being driven. If one assumes the pressure, P, is uniform over the bottom surface of the membrane then the net force on any element of the area ΔS of the membrane due to the pressure is

$$(P - P_0)\Delta S = P_1 \Delta A \cos\omega t$$

Adding this force to the tensile forces and writing down Newton's second law for the element of area ΔS one obtains after passing to the limit the following wave equation

$$c^2 \left[\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \phi^2} \right] + \frac{P_1}{\sigma} \cos t = \frac{\partial^2 z}{\partial t^2} \qquad (5.20)$$

Any function $z(r,\phi,t)$ describing the motion of the membrane under the above conditions must satisfy the wave equation. Now experimentally it is found that under the above conditions, the membrane reaches a steady state in which each portion is vibrating harmonically with the same frequency, ω, as that of the oscillator. This suggests there must exist a solution of (5.20) of the form

$$z(r,\phi,t) = \psi(r,\phi) \cos(\omega t + \beta)$$

Substituting this in (5.18) one obtains after expanding the $\cos(\omega t + \beta)$ term, and rearranging

$$\left[c^2\left\{\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right\} - \omega^2\psi\right\}\cos\beta + \frac{P_1}{\sigma}\Bigg]\cos\omega t$$

$$- \left[\left\{c^2\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right] - \omega^2\psi \quad \sin\beta\right]\sin\omega t = 0$$

This condition must hold for all times, a requirement that can be satisfied if the coefficient of $\sin\omega t$ and $\cos\omega t$ is zero. Both coefficients will be zero if $\beta = 0$ and

$$c^2\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r}\frac{\partial^2\psi}{\partial\phi^2}\right] - \omega^2\psi = -\frac{P_1}{\sigma} \qquad (5.21)$$

~~For~~ If P_1/σ is zero, the above equation has the solution

$$\psi(r,\phi) = J_m(kr)\left[A'_m\cos m\phi + B'_m\sin m\phi\right]$$

where m = 0,1,2 ... Hence (5.19) has a solution

$$\psi(r,\phi) = J_m(kr)\left[A'_m\cos m\phi + B'_m\sin m\phi\right] - \frac{P_1}{\sigma\omega^2}$$

and there exists a solution of the wave equation (5.18) of the form

$$z(r,\phi,t) = \left\{J_m(kr)\left[A'_m\cos m\phi + B'_m\sin m\phi\right] - \frac{P_1}{\sigma\omega^2}\right\}\cos\omega t$$

If this is to satisfy the boundary condition that $z(a,\phi,t) = 0$ one must have

$$A'_m\cos m\phi + B'_m\sin m\phi = \frac{P_1}{\sigma\omega^2 J_m(ka)}$$

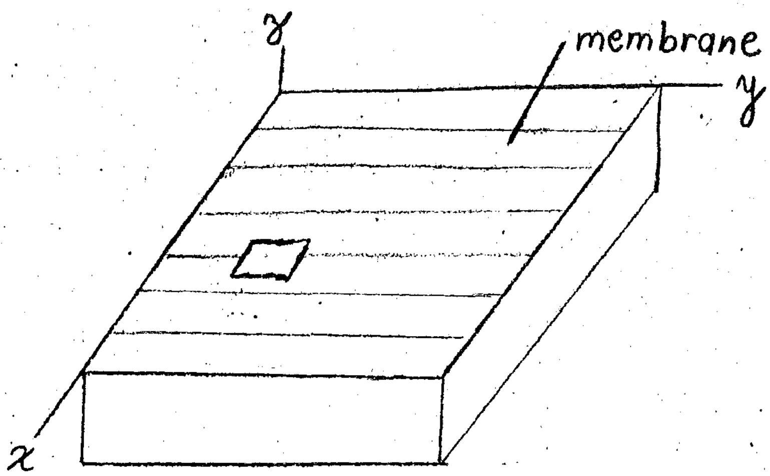
This can be satisfied only if m = 0 and

$$A'_o = \frac{P_1}{\sigma\omega^2 J_o(ka)}$$

so that a solution of (5.20) which satisfies the boundary condition becomes

$$z(r,t) = \frac{P_1}{\sigma\omega^2}\left[\frac{J_0(\frac{\omega}{c}r)}{J_0(\frac{\omega}{c}a)} - 1\right] \cos\omega t$$

This expression predicts infinitely large amplitudes at those frequencies for which $J_0(\omega a/c) = 0$. These frequencies correspond to the eigen frequencies for $m = 0$. A more realistic wave equation for the driven membrane would include damping forces and the corresponding solutions would not show these discontinuities. However, one would still expect relatively large amplitudes to occur at or near the characteristic frequencies.



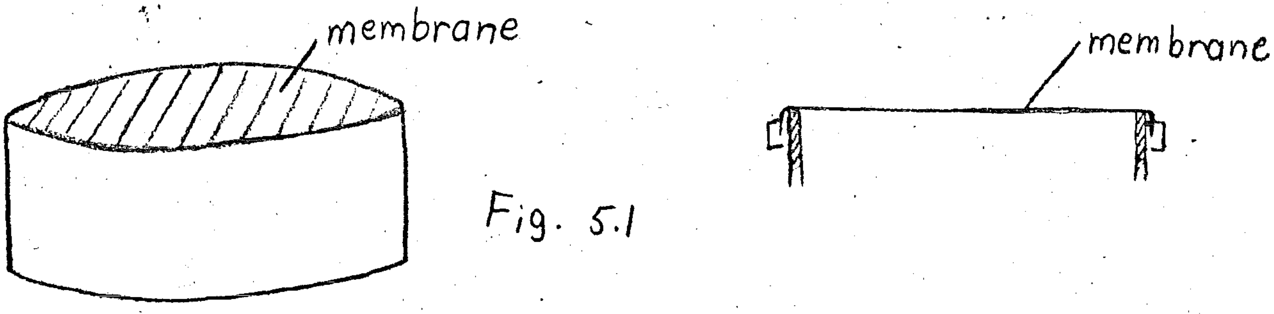


Fig. 5.1

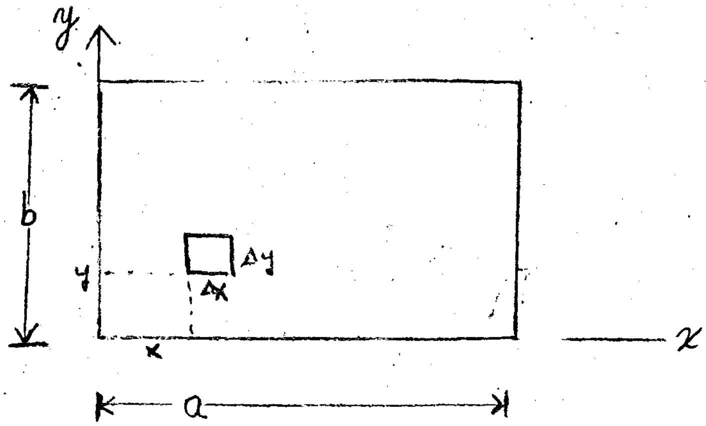
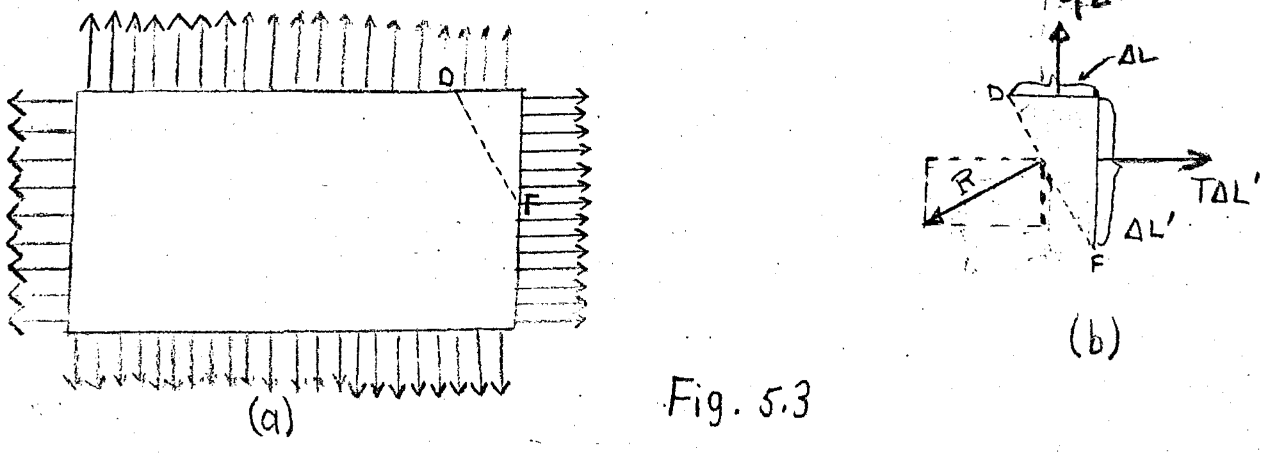


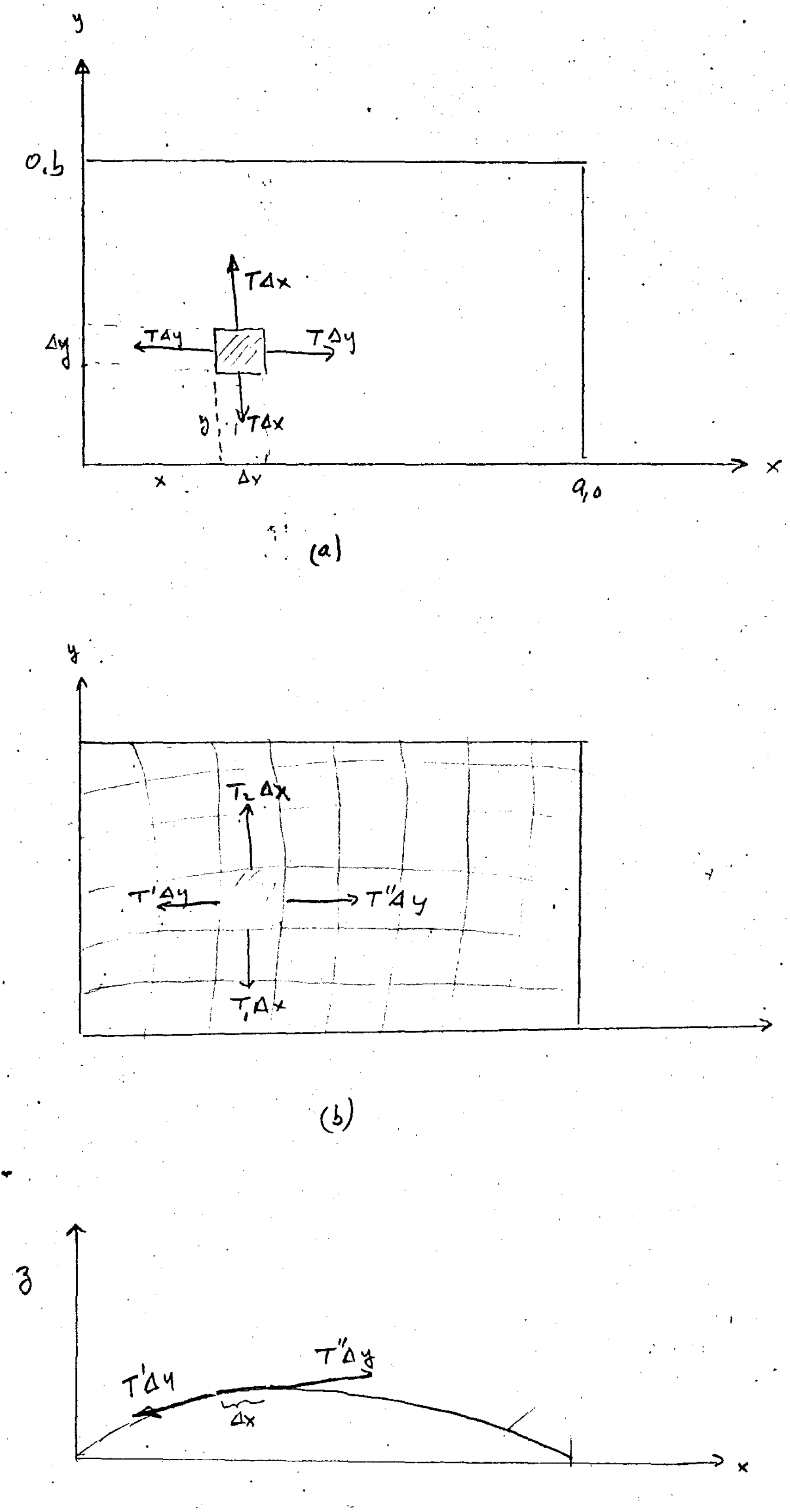
Fig. 5.2



(a)

(b)

Fig. 5.3

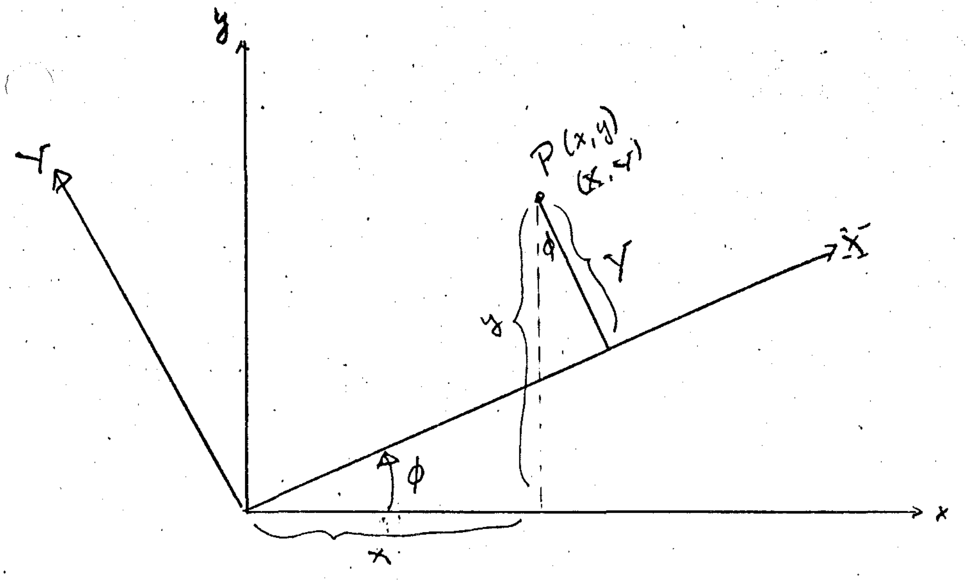


(a)

(b)

(c)

Fig 5.4



$$x = X\cos\phi - Y\sin\phi \qquad X = x\cos\phi + y\sin\phi$$

$$y = X\sin\phi + Y\cos\phi \qquad Y = y\cos\phi - x\sin\phi$$

Fig 5.5

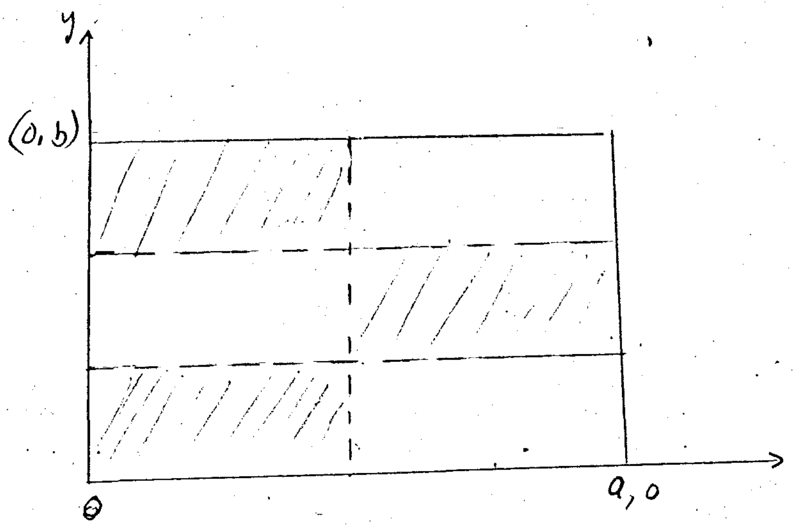


Fig 5.6

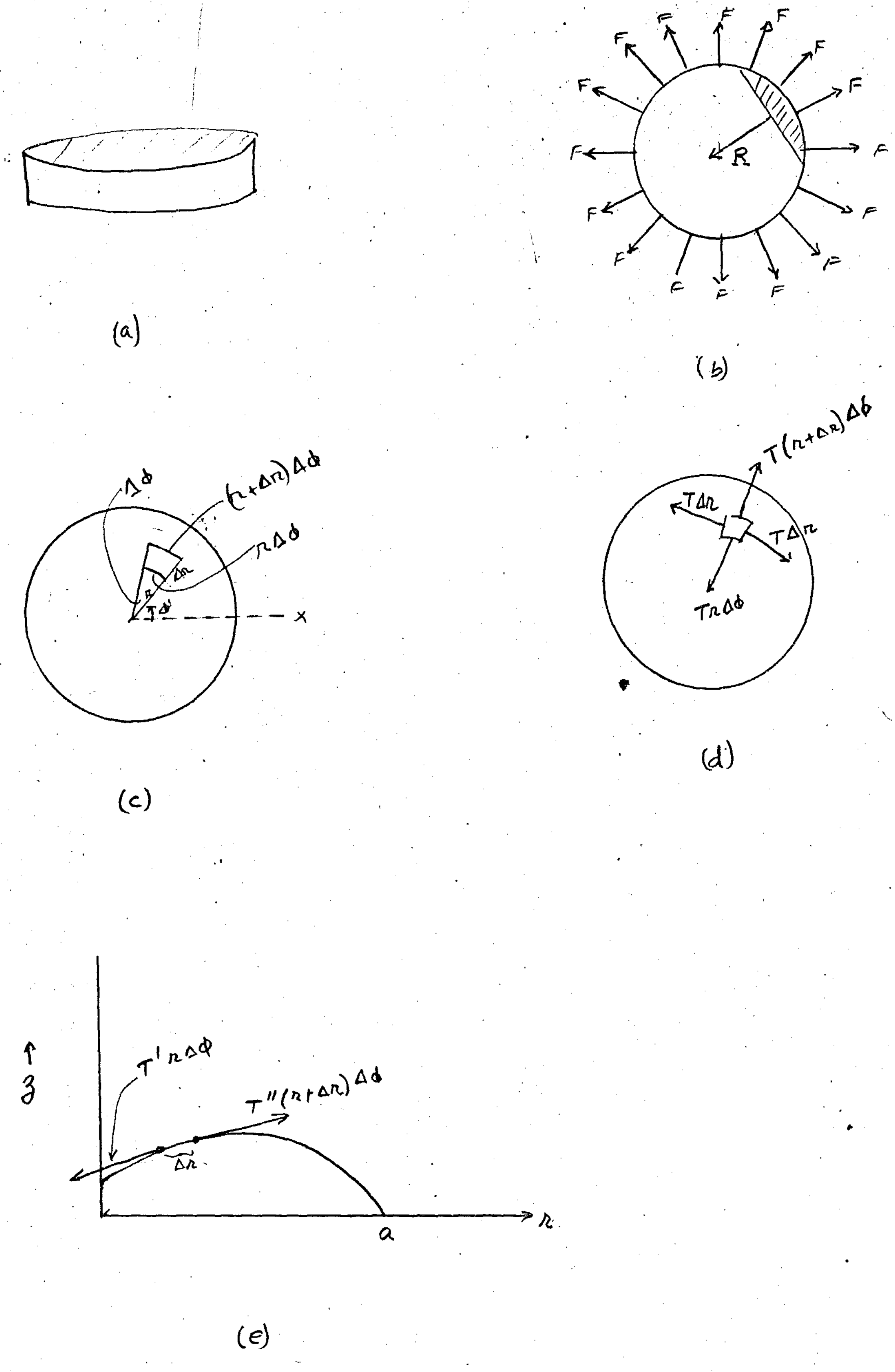
(a)
F
R
(b)
Δϕ
(r+Δr)Δϕ
rΔϕ
Δr
r
Δϕ
x
(c)
T(r+Δr)Δϕ
TΔr
TΔr
TrΔϕ
(d)
z
T′rΔϕ
T″(r+Δr)Δϕ
Δr
a
r
(e)

Fig 5,7

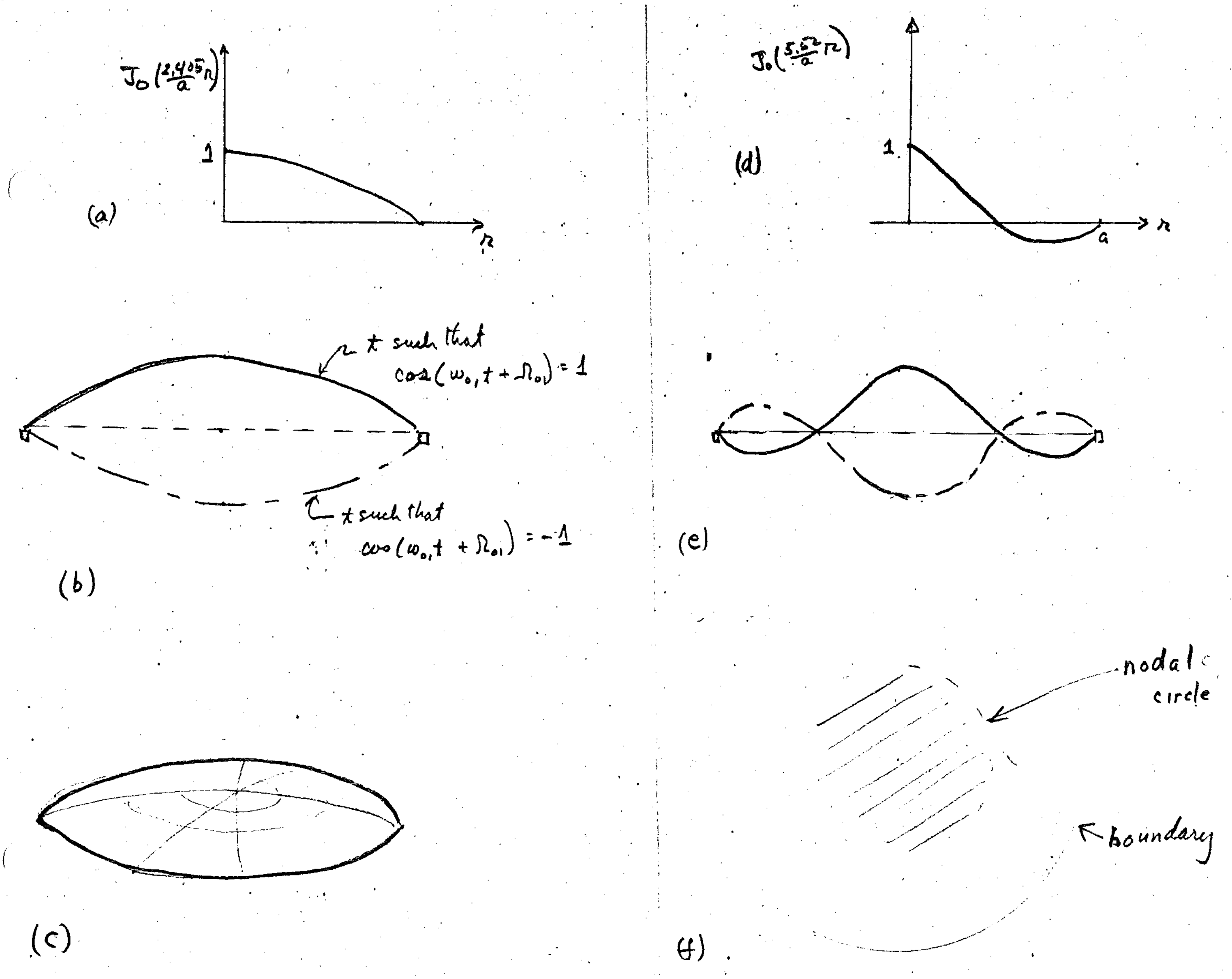

$J_0(\frac{2.405}{a}r)$
1
(a)
r
t such that $\cos(\omega_{01}t+\Omega_{01}) = 1$
t such that $\cos(\omega_{01}t+\Omega_{01}) = -1$
(b)
(c)
$J_0(\frac{5.52}{a}r)$
1
(d)
a
r
(e)
nodal circle
boundary
(f)

Fig 5.9

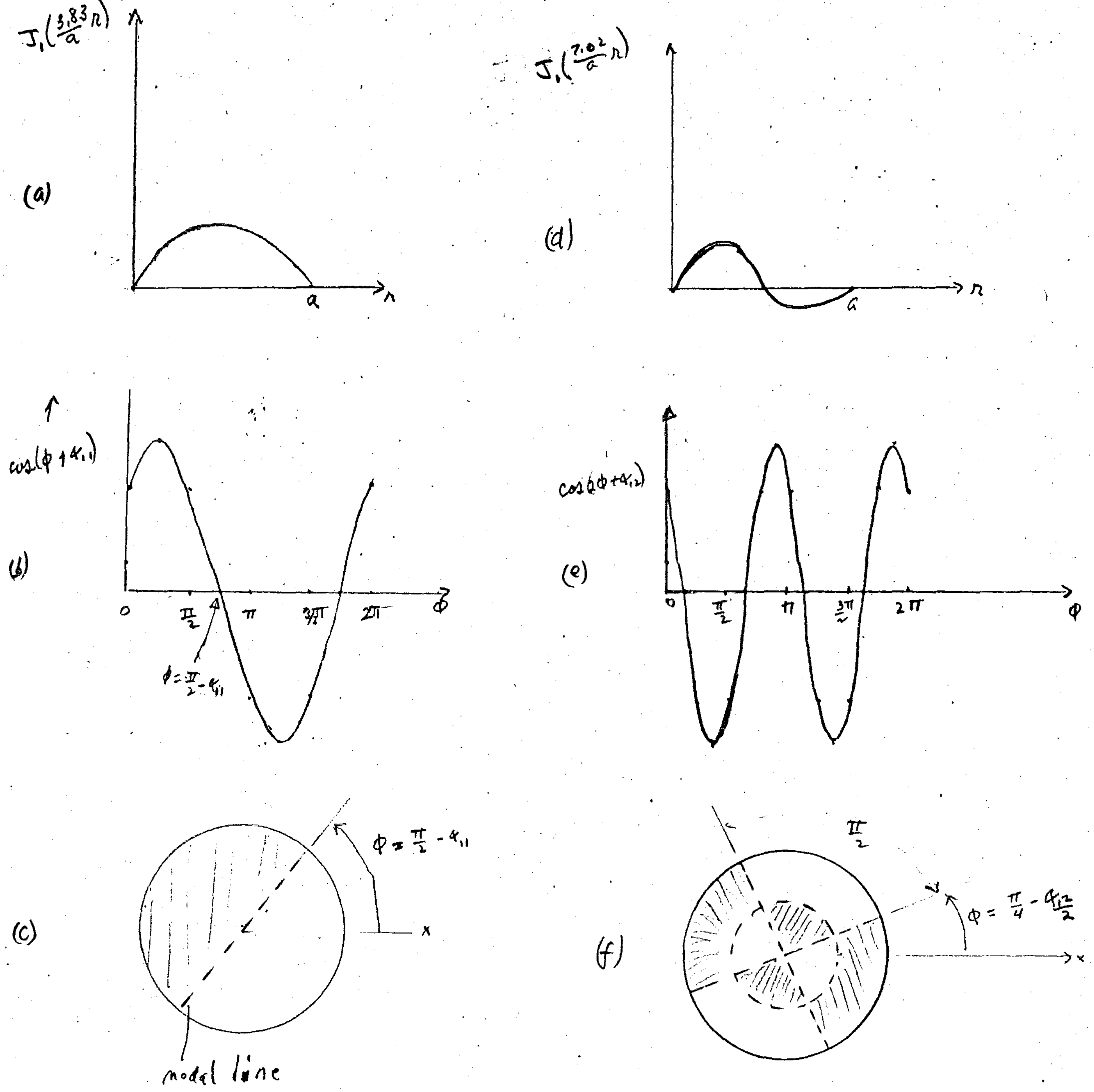

$J_1(\frac{3.83}{a}r)$
(a)
a
r
$\cos(\phi+\alpha_{11})$
(b)
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
ϕ
$\phi=\frac{\pi}{2}-\alpha_{11}$
(c)
$\phi=\frac{\pi}{2}-\alpha_{11}$
x
nodal line
$J_1(\frac{7.02}{a}r)$
(d)
a
r
$\cos(2\phi+\alpha_{12})$
(e)
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
ϕ
(f)
$\frac{\pi}{2}$
$\phi=\frac{\pi}{4}-\frac{\alpha_{12}}{2}$
x

Fig 5.10

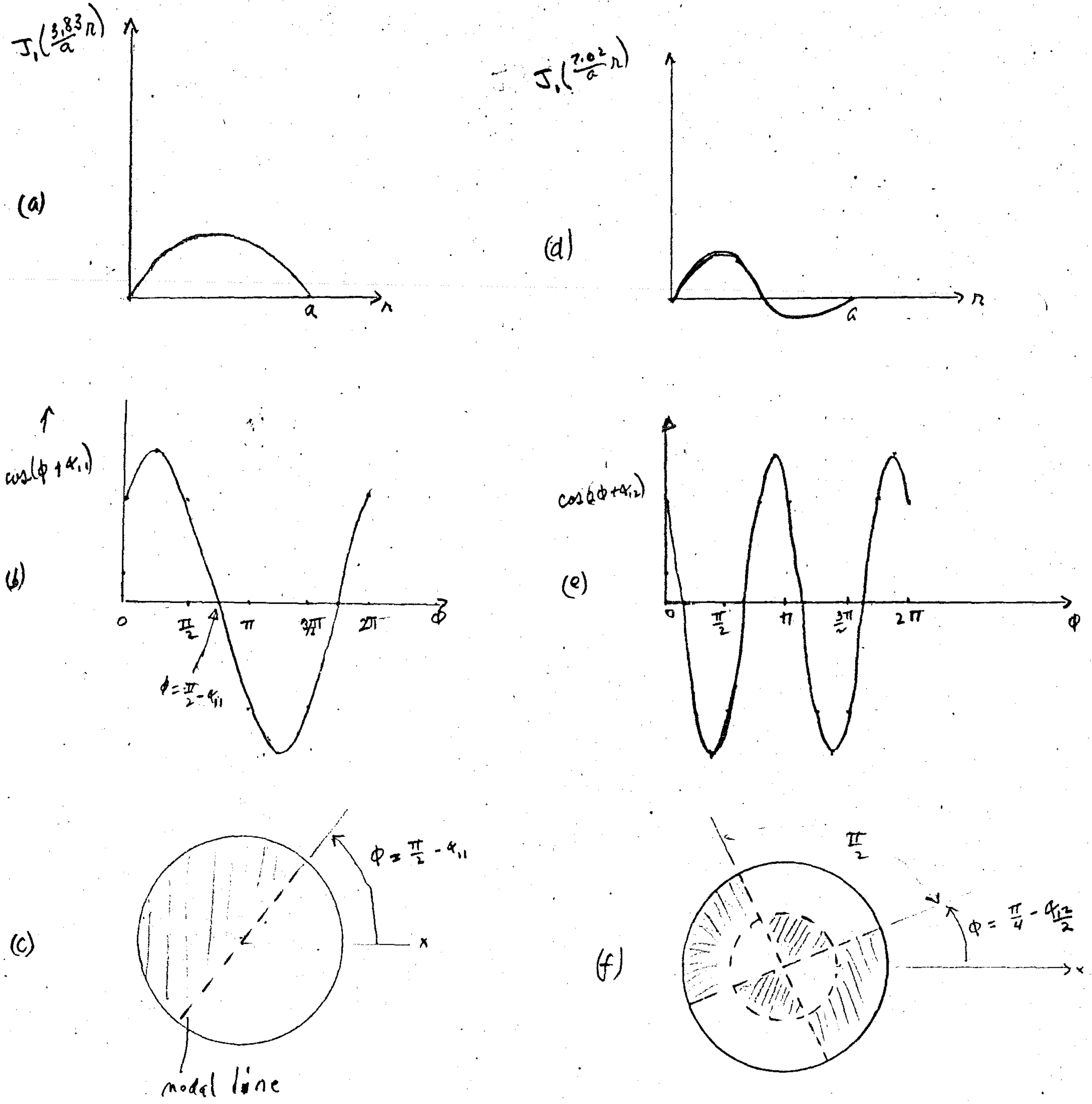
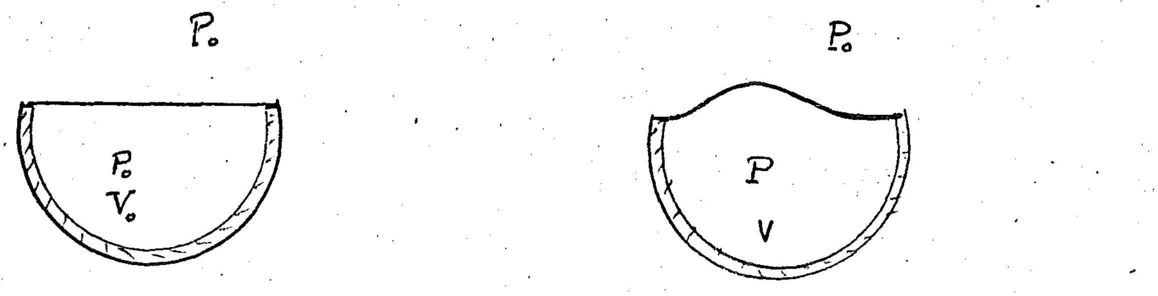

$J_1(\frac{3.83}{a}r)$
(a)
a
r
$J_1(\frac{7.02}{a}r)$
(d)
a
r
$\cos(\phi+\alpha_{11})$
(b)
0
$\frac{\pi}{2}$
π
$3\pi/2$
2π
ϕ
$\phi=\frac{\pi}{2}-\alpha_{11}$
$\cos(2\phi+\alpha_{12})$
(e)
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π
ϕ
(c)
$\phi=\frac{\pi}{2}-\alpha_{11}$
x
nodal line
(f)
$\frac{\pi}{2}$
$\phi=\frac{\pi}{4}-\frac{\alpha_{12}}{2}$
x

Fig 5.10



a) Membrane at rest

b) time t with membrane in motion

Fig 5.11

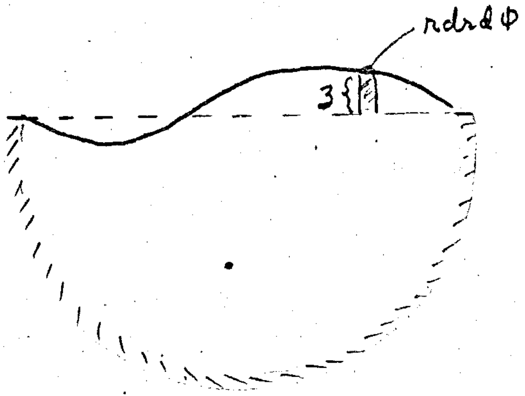


Fig 5.12

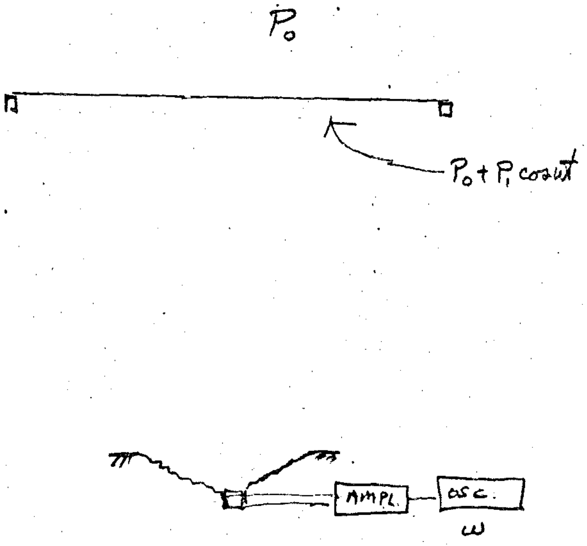
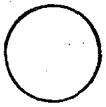
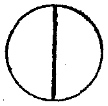
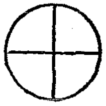
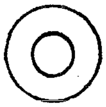
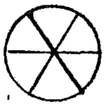
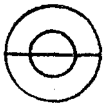
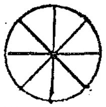
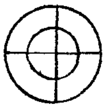
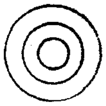
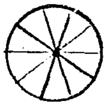


Fig 5,13

Table 5.2 Nodal Patterns and Frequency Relations for Ten Smallest Characteristic Frequencies of Membrane with Circular Boundary

CHARACTERISTIC FREQ	Ratio to FUNDAMENTAL	NODAL PATTERN	RADII OF NODAL CIRCLES
f_{01}	1.000		
f_{11}	1.593		
f_{21}	2.135		
f_{02}	2.295		$r = 0.437a$
f_{31}	2.653		
f_{12}	2.917		$r = 0.543\ a$
f_{41}	3.155		
f_{22}	3.500		$r = 0.610a$
f_{03}	3.598		$r_1 = 0.278a$ $r_2 = 0.638a$
f_{51}	3.648		

Chapter VI. WAVES IN FLUIDS

For longitudinal waves in a thin rod, the displacement of any given element of the rod has only a single component, and hence a single coordinate, ξ, is sufficient to describe this displacement. Similarly, the displacement of any element of a membrane has only a single component and a single coordinate z is sufficient to describe this displacement. The displacement of an element of a fluid has in general <u>three</u> components. In addition, <u>three</u> coordinates, say x, y and z, are required to locate an element, as contrasted to two for an element of a membrane, and one for an element of a rod. Moreover, one often prefers to describe waves in fluids in terms of quantities other than those of the displacement. For these and other reasons, the description of waves in fluids is more complicated. None the less, the derivation of the wave equation follows along the same general lines; one uses the stress-strain relations and requires that the motion of each element be governed by Newton's second law. The type of waves which are propagated in fluid are called "compressional" or "dilational" or "longitudinal" or "sound" waves.

6.1 Wave Equation for Waves in Fluids

Consider a confined fluid as indicated in Fig. 6.1a. By an element of the fluid (also referred to as a particle) one means a tiny portion of the fluid. To be more specific let the element located at point M (x,yz) to be the <u>mass</u> of fluid contained in a tiny cubical volume located at M, as indicated in Fig. 6.1b. If the external force $\vec{F}_{ext}$ of Fig. 6.1a is changed to a new value, then after equilibrium has been established, the element of fluid originally at M in general will be at some new location and the dimensions of the element will have changed. Let the x, y and z components of the displacement undergone by point M be ξ, η and ζ respectively. We assume that the displacement of point M is a suitable measure of the displacement of the <u>element</u>, and as we learned in Chapter 1 the change in shape of the element can be determined from $\frac{\partial \xi}{\partial x}$, $\frac{\partial \xi}{\partial y}$ and $\frac{\partial \xi}{\partial z}$ all evaluated at point M. In the static case the relation between a change in pressure and the change in the shape of the element was given by (1.6), namely

$$\Delta P = -B\left[\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right]$$

where B is the bulk modulus of the fluid.

If the force F_{ext} of Fig. 1.6a is varied rapidly about some mean value, then in general the pressure at any instant of time will be different at different points of the fluid and at a point such as M will vary rapidly above and below some mean value P. If P' is the (instantaneous) pressure at M at any time t one assumes that

$$P' - P = -B\left[\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right]$$

i.e., that the static relationship holds at every instant of time. If one defines the acoustic pressure $\mathcal{P}$ at a point as the difference between the instantaneous pressure <u>P</u>' and the mean or equilibrium pressure P, i.e.,

$$\mathcal{P} = P' - P \tag{6.1}$$

the above relationship becomes

$$\mathcal{P} = -B\left[\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right] \tag{6.2}$$

It is worth noting that the acoustic pressure is an <u>algebraic</u> quantity while P' and P are not. Also, for most cases of interest, the pressure changes are sufficiently rapid so that the appropriate modulus is the <u>adiabatic</u> bulk modulus.

The forces acting on the element of fluid at any instant are those due to the pressure at the six faces of the small cubical volume containing the element. Considering only the x-equation of motion one has (see Fig. 6.2).

$$\left[\underline{P}'\ (x,y,z,t) - \underline{P}'(x + \Delta x,\ y,\ z,\ t)\right] \Delta y\ \Delta z = \rho \Delta x\, \Delta y\, \Delta z\, \frac{\partial^2 \xi}{\partial t^2}$$

where $\underline{P}'\ (x,y,z,t)$ and $P'\ (x + \Delta x,\ y,\ z,\ t)$ are the instantaneous pressures at faces ABCD and EFGH respectively, and ρ is the density of fluid. Dividing by $\Delta x\ \Delta y\ \Delta z$ and passing to the limit one has

$$-\frac{\partial P'}{\partial x} = \rho \frac{\partial^2 \xi}{\partial t^2}$$

or in terms of the acoustic pressure

$$-\frac{\partial p}{\partial x} = \rho \frac{\partial^2 \xi}{\partial t^2} \qquad (6.3)$$

Similarly for the y and z equations of motion one gets

$$-\frac{\partial p}{\partial y} = \rho \frac{\partial^2 \eta}{\partial t^2}$$

$$-\frac{\partial p}{\partial z} = \rho \frac{\partial \zeta}{\partial t^2} \qquad (6.4$$

Differentiating (6.2) twice with respect to time and interchanging the order of differentiation ~~as~~ ON the right side one obtains

$$\frac{\partial^2 p}{\partial t^2} = -B\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 \xi}{\partial t^2}\right) + \frac{\partial}{\partial y}\left(\frac{\partial^2 \eta}{\partial t^2}\right) + \frac{\partial}{\partial z}\left(\frac{\partial^2 \zeta}{\partial t^2}\right)\right]$$

Substituting from (6.3) one obtains the wave equation

$$\boxed{c^2\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\right] = \frac{\partial^2 p}{\partial t^2} \qquad c = \sqrt{B/\rho}} \qquad (6.5)$$

for waves in fluids.

6.2 Plane Waves, Velocity of Propagation

Although the wave equation (6.5) is different from any encountered thus far it should be evident that any function ($P(\nu)$) where $\nu = x \pm ct$ or $y \pm ct$ or $z \pm ct$ would satisfy it, since if $P(\nu)$ is a function only of <u>one</u> of the coordinates, the wave equation reduces to the form for waves on strings. Such functions represent what are called plane waves. The real 'part' of a function like

$$\mathcal{P}(x,t) = \underset{\sim}{A}\, e^{i(\omega t-kx)} \qquad k = \omega/c$$

for example represents a plane harmonic wave being propagated in the +x direction. It is called a <u>plane</u> wave since the pressure is independent of y and z and hence at any instant of time is the same at all points of any plane perpendicular to the x-axis.

It is not difficult to show following the method used in section 3.3, that any function $\mathcal{P}(\nu)$ where

$$\nu = x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta \qquad (6.6)$$

will also satisfy the wave equation (6.5) for arbitrarily chosen values of θ and ϕ. By choosing a new coordinate system, X,Y,Z such that the direction cosines of the +X-axis with respect to the xyz coordinate system are $\sin\theta \cos\phi$, $\sin\theta \sin\phi$ and $\cos\theta$ respectively, as indicated in Fig. 6.3a, such functions can be written $\mathcal{P}(X-ct)$, and thus represent <u>plane</u> waves being propagated in the +X direction with a velocity c. For example, the function

$$\mathcal{P}(x,y,t) = \underset{\sim}{A}\, e^{i[\omega t-k(x \cos\phi + y \sin\phi)]}$$

where $k = \omega/c$ represents a plane harmonic wave being propagated in the +X direction where the +X-axis makes an angle ϕ with the +x-axis as indicated in Fig. 6.3b. Note that for such a wave, the acoustic pressure at any instant of time is the same at all points of any plane perpendicular to the X-axis, and that these planes are in this instance parallel to the z-axis.

The speed c at which any plane wave is propagated in any fluid is given by

$$c = \sqrt{B/\rho}$$

where B_a is the adiabatic bulk modulus, and ρ is the density of the fluid. For an ideal gas it can be shown that for small variations of the pressure about some equilibrium pressure P_0, the adiabatic bulk modulus

$$B_a = \gamma P_0$$

where γ is the ratio of the specific heat at constant P to that at constant volume. Thus for an ideal gas

$$c = \sqrt{\frac{\gamma P_0}{\rho}}$$

This result correctly predicts the speeds of propagation of plane waves in real gases at ordinary pressures. Also for n moles of ideal gas of mass m, and molecular weight M

$$PV = nRT; \quad V = \frac{m}{\rho} = \frac{nM}{\rho}; \quad \frac{P}{\rho} = \frac{RT}{M}$$

so that

$$c = \sqrt{\frac{\gamma RT}{M}} = \text{const} \sqrt{T}$$

Experimental results on real gases at ordinary pressures bear out this prediction that the speed of propagation is proportional to the square root of the absolute temperature. The speed of sound in air at 0°C is 331.6 m/sec and this increases approximately 0.6 m/sec per degree rise in temperature.

The velocity of propagation of plane waves in liquids is for the most part higher than that in gases, the velocity of sound in water being 1480 m/sec at 20°, a figure about 4 times the speed of sound in air. The speed also increases with the temperature, although there is no simple relationship as is the case with gases. Table 61 gives the speed of sound in some of the more common gases and liquids.

6.3 Harmonic Solutions of the Wave Equation

Following the usual procedure for finding solutions to partial differential equations one looks for solutions of (6.5) of the form

$$P(x,y,z,t) = X(x)\ Y(z)\ Z(z)\ H(t)$$

Substituting into (6.5) leads to the requirement that for all x,y,z and t

$$c^2 \left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}\right] = \frac{1}{H}\frac{d^2H}{dt^2}$$

a condition that requires both sides equal a constant, say $-\omega^2$. One thus obtains

$$\frac{d^2H}{dt^2} = -\omega^2 H \qquad (6.7)$$

and

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 - \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2}$$

where $k = \omega/c$. Once again this second equation can only be satisfied for all values of x, y and z if both sides equal a constant say $-\alpha^2$ which leads to

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -(k^2-\alpha^2) - \frac{1}{Z}\frac{d^2Z}{dz^2} \qquad (6.8)$$

The second of these two equations can only be satisfied for all values of x and y only if both sides equal a constant say $-\beta^2$.

Thus

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2 \tag{6.9}$$

and

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -(k^2-\alpha^2-\beta^2) \tag{6.10}$$

Solutions of (6.7), (6.8), (6.9) and (6.10) are readily apparent if $k^2 > \alpha^2 + \beta^2$. Setting $\gamma^2 = k^2 - \alpha^2 - \beta^2$, a solution of the wave equation is

$$P(x,y,z,t) = (a_1\cos\alpha x+b_1\sin\alpha x)(a_2\cos\beta y+b_2\sin\beta y)(a_3\cos\gamma z+b_3\sin\gamma z)(a_4\cos\omega t+b_4\sin\omega t) \tag{6.11}$$

This is a solution for all positive values of α, β, and γ and ω and for arbitrary values of the constants $a_1 \ldots a_4$, $b_1 \ldots b_4$. Note that if such a function does describe the pressure wave in a fluid, the acoustic pressure at any point varies harmonically in time with a frequency ω.

Using trig identities the harmonic solution (6.11) can be recast in the following form

$$\begin{aligned}P(x,y,z,t)=A\{&\cos(\alpha x+\beta y+\gamma z-\omega t+\Omega_1) + \cos(\alpha x+\beta y+\gamma z+\omega t+\Omega_2)\\ &+ \cos(\alpha x+\beta y-\gamma z-\omega t\pm\Omega_3) + \cos(\alpha x+\beta y-\gamma z+\omega t+\Omega_4)\\ &+ \cos(\alpha x-\beta y+\gamma z-\omega t+\Omega_5) + \cos(\alpha x-\beta y+\gamma z+\omega t\pm\Omega_6)\\ &+ \cos(\alpha x-\beta y-\gamma z-\omega t+\Omega_7) + \cos(\alpha x-\beta y-\gamma z+\omega t+\Omega_8)\}\end{aligned} \tag{6.12}$$

Each one of the eight terms in this expression is of the form $P(\nu)$ where ν is given by (6.6), and thus represents a plane harmonic wave being propagated in a direction determined by the values of α, β, and γ. The direction of propagation is in general different for each wave. For example, the direction of propagation of the plane wave represented by the first term is along a line whose direction cosines with respect to the x,y,z coordinate system are $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta$ where $\tan\theta = \sqrt{\alpha^2+\beta^2}/\gamma$ and $\tan\phi = \beta/\alpha$, while the direction of propagation of the plane wave represented by the third term is along a line whose direction cosines are $\sin\theta'\cos\phi$, $\sin\theta'\sin\phi$, $\cos\theta'$ where $\theta' = \pi - \theta$.

6.4 Boundary Conditions, Eigen Frequencies

Suppose the fluid is confined by a rigid vessel in the form of a box of length L_x width L_y and height L_z as indicated in Fig. 6.4. Any particle of fluid in contact with the face OMNQ is prevented by the wall from moving in the x direction, i.e.

$$\xi(0,y,z,t) = 0$$

and consequently

$$\left.\frac{\partial^2\xi}{\partial t^2}\right|_{0,y,z,t} = 0$$

If this latter condition is satisfied it follows from (6.3) that

$$\left.\frac{\partial P}{\partial x}\right|_{0,y,z,t} = 0$$

An harmonic solution of the form (6.10) can be made to satisfy this condition by setting $b_1=0$. Similarly at the face opposite DMNQ the particle displacement is zero, i.e.

$$\xi(L_x,y,z,t) = 0 \Rightarrow \left.\frac{\partial P}{\partial x}\right|_{L_x,y,z,t} = 0$$

The harmonic solution (6.11) with $b_1=0$ will satisfy the above condition provided

$$\alpha = n_x\pi/L_x \qquad n_x = 0,1,2 \dots \dots \tag{6.13}$$

Similarly the boundary conditions

$$\eta(x,0,z,t) = 0 \Rightarrow \left.\frac{\partial P}{\partial y}\right|_{x,0,z,t} = 0$$

$$\eta(x,0,z,t)=0 \implies \left.\frac{\partial P}{\partial y}\right|_{x,0,z\,t}=0$$

can be met by a function of form (6.11) by choosing $b_2=0$ and

$$\beta=\frac{n_y\pi}{L_y} \qquad n_y = 0,1,2,3\ \ldots \qquad (6.14)$$

and the boundary conditions

$$\zeta(x,y,0,t)=0 \implies \left.\frac{\partial P}{\partial z}\right|_{x,y,0,t}=0$$

$$\zeta(x,y,L_z,t)=0 \implies \left.\frac{\partial P}{\partial z}\right|_{x,y,L_z,t}=0$$

can be met by choosing $b_2=0$ and

$$\gamma=\sqrt{k^2-(\alpha^2+\beta^2)}=n_z\pi/L_z \qquad n_z=0,1,2,3\ \ldots \qquad (6.15)$$

Thus the harmonic solution

$$P(x,y,z,t)=\cos\frac{n_x\pi}{L_x}x\,\cos\frac{n_y\pi}{L_y}y\,\cos\frac{n_z\pi}{L_z}z\left[A_{n_xn_yn_z}\cos\omega_{n_xn_yn_z}t+B_{n_xn_yn_z}\sin\omega_{n_xn_yn_z}t\right]$$

where

$$\omega_{n_xn_yn_z}=\pi c\sqrt{\left(\frac{n_x}{L_x}\right)^2+\left(\frac{n_y}{L_y}\right)^2+\left(\frac{n_z}{L_z}\right)^2} \qquad \begin{array}{l} n_x=0,1,2,3 \\ n_y=0,1,2,3 \\ n_z=0,1,2,3 \end{array} \qquad (6.17)$$

satisfies both the wave equation and the boundary conditions. The latter expression which gives the eigen frequencies is determined from (6.15), (6.13) and (6.14). If L_x is the largest dimension of the box, the smallest of the eigen frequencies is

$$\omega_{100}=\pi c/L_x \qquad f_{100}=c/2L_x$$

and the corresponding eigen function is

$$P_{100}(x,t)=\cos\frac{\pi}{L}x\left[A_{100}\cos\frac{\pi c}{L_x}t+B_{100}\sin\frac{\pi c}{L_x}t\right]$$

$$P_{100}(x,t) = C_{100}\cos\frac{\pi}{L_x}x\left[\cos\frac{\pi c}{L_x}t+\Omega_{100}\right]$$

Thus if the system is vibrating in its fundamental mode, the acoustic pressure amplitude is a maximum at x=0 and x=L and is zero at x=L/2. It should be evident that the above expression can be recast so as to represent two plane waves, both propagated with a velocity c, one in the +x direction and one in the -x direction. For either of the characteristic modes corresponding to ω_{010} and ω_{001}, the situation is similar; the pressure amplitude is maximum at the two opposite faces and zero in the middle, and the pressure variations can be thought of as being due to two plane waves moving in opposite directions. In fact for all characteristic modes for which only one of the n's is different from zero, the pressure waves are plane waves. For a characteristic mode corresponding to $n_x = 1$, $n_y = 1$, $n_z = 0$, the eigen function is

$$P_{110}(x,y,t) = C_{110}\cos\frac{\pi}{L_x}x\cos\frac{\pi}{L_y}y\cos(\omega_{110}t+\Omega_{110})$$

where $\omega_{110} = \pi c\sqrt{(1/L_x)^2 + (1/L_y{}^2)}$. This mode has nodal planes at $x = L_x/2$ any $y = L_y/2$. Higher modes have progressively more and more nodal planes.

The sum of all the characteristic modes

$$P(x,y,z,t) = \sum_{n_x=0}^{\infty}\sum_{n_y=0}^{\infty}\sum_{n_z=0}^{\infty}\cos\frac{n_x\pi}{L_x}x\cos\frac{n_y\pi}{L_y}y$$

$$\cos\frac{n_z\pi}{L_z}z\left[A_{n_xn_yn_z}\cos w_{n_xn_yn_z}t + B_{n_xn_yn_z}\sin w_{n_xn_yn_z}t\right]$$

is also a solution satisfying the boundary conditions and can if desired by a proper choice of the constants $A_{n_x n_y n_z}$ and $B_{n_x n_y n_z}$ be made to fit a set of initial conditions.

6.5 Propagation in a Rectangular Wave Guide

If one of the dimensions, say L_z of the box of Fig. 6.4, is made indefinitely large, one has what is called a rectangular wave guide. The boundary conditions at the four walls of the guide are, of course, the same as they are for the closed box. It follows that

$$p(x,y,z,t) = \left[a_z \cos \gamma z + b_z \sin \gamma z\right] \cos \frac{n_x \pi}{L_x} x \cos \frac{n_y \pi}{L_y} y \left[A \cos \omega t + B \sin \omega t\right]$$

$$= C \cos(\gamma z + \delta) \cos \frac{n_x \pi}{L_x} x \cos \frac{n_y \pi}{L_y} y \cos(\omega t + \Omega). \qquad (6.18)$$

where

$$\gamma = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n_x \pi}{L_x}\right)^2 + \left(\frac{n_y \pi}{L_y}\right)^2}$$

is an harmonic solution of the wave equation satisfying the boundary conditions for any integer values of n_x and n_y and for any value ω for which γ is real. For any <u>fixed</u> value of ω there is a harmonic solution like (6.18) for each pair of values of n_x and n_y for which

$$\left(\frac{\omega}{c}\right)^2 > \frac{n_x^2 \pi^2}{L_x^2} + \frac{n_y^2 \pi^2}{L_y^2} \qquad (6.19)$$

Suppose a harmonic solution of the form (6.18) did actually describe the acoustic pressure at all points of the guide, and one made measurements of the acoustic pressure at points along a line parallel to z-axis Since every point on this line has the same x and y coordinate, say x_1 and y_1, for points on this line (6.18) could be recast in the form

$$P(x_1, y_1, z, t) = \left[\frac{C}{2}\cos\frac{n_x\pi}{L_x}x_1\cos\frac{n_y\pi}{L_y}y_1\right]\left[\cos(\gamma z - \omega t + \Omega - \delta) + \cos(\gamma z + \omega t + \Omega + \delta)\right]$$

$$= A'\cos\gamma\left(z - \frac{\omega}{\gamma}t + \Omega - \delta\right) + A'\cos\gamma\left(z + \frac{\omega}{\gamma}t + \Omega + \delta\right)$$

where A' is a constant standing for the first bracket.

From its appearance, one could argue that the first term represents a wave being propagated in the +z-direction and the second term a wave being propagated in the -z-direction, both waves being propagated with a speed

$$c' = \frac{\omega}{\gamma} = \frac{\omega}{\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n_x\pi}{L_x}\right)^2 - \left(\frac{n_y\pi}{L_y}\right)^2}} = \frac{c}{\sqrt{1 - \left[\left(\frac{n_x\pi}{L_x}\right)^2 + \left(\frac{n_y\pi}{L_y}\right)^2\right]\left(\frac{c\pi}{\omega}\right)^2}}$$

One thus interprets harmonic solutions such as (6.18) as representing waves being propagated along the guide. For any fixed value of ω there is a solution of the form (6.18) for each pair of values of n_x and n_y which satisfy the restriction (6.19). Each such solution is referred to as a mode, the 00 mode being

$$P_{00}(z, t) = C_{00}\cos\left(\frac{\omega}{c}z + \delta\right)\cos(\omega t + \Omega)$$

$$= \frac{C_{00}}{2}\left\{\cos k\left(z - ct + \frac{\delta - \Omega}{k}\right) + \cos k\left(z + ct + \frac{\delta + \Omega}{k}\right)\right\}$$

This expression represents two plane waves, each propagated with a velocity $c = \sqrt{B/\rho}$. The 01 mode,

$$P_{01}(x, z, t) = C_{01}\cos\frac{\pi}{L_x}x\cos(\gamma z + \delta)\cos(\omega t + \Omega)$$

is not a plane wave and its speed of propagation down the wave guide is

$$c' = c\Big/\sqrt{1-\left(\frac{\pi c}{\omega L_x}\right)^2}$$

Note that this velocity, which is referred to as the <u>phase</u> velocity is <u>greater</u> than c. All modes except the 00 mode have phase velocities greater than c, and none are plane waves.

In general, if an harmonic source (e.g. a loud speaker) of frequency ω is located at some point of a wave guide, one expects that some time after the source is started, the acoustic pressure at any point in the guide will be given by some combination of the allowed modes. For any source frequency it is possible by virtue of condition (6.19) to choose the dimensions of the wave guide to insure that only the 00 mode will be present, and thus that the waves in the guide will be plane waves. As may be verified from (6.19) for a square wave guide 0.15 m x 0.15m containing air at 20°C only the 00 mode will be present for all source frequencies below 1140 hertz. In many cases of interest, the dimensions of the wave guides are such that one has to deal only with plane waves.

<u>6.6 Particle Velocity, Specific Acoustic Impedance for Plane Waves</u>

Plane harmonic waves in fluids are an important special case and the remainder of this chapter will deal exclusively with such waves. The real part of

$$\underset{\sim}{P}(x,t) = \underset{\sim}{A}\, e^{\underset{\sim}{i}(\omega t-kx)} + \underset{\sim}{B}\, e^{\underset{\sim}{i}(\omega t+kx)} \qquad (6.20)$$

where $k=\omega/c$ represents two plane waves being propagated in the + and -x-directions respectively. If such waves existed in a fluid, one could find how the acoustic pressure at any point of the fluid varies with time merely by inserting the x-coordinate of the point into (6.20). To find the displacement of the element of fluid (i.e. the particle) located at that point as a function of time, one makes use of equations (6.3) and (6.4) which relate the components ξ, η, and ζ of the particle displacement at any point to the pressure gradient at that point. For the wave represented by (6.20) one has from (6.3)

$$-\frac{\partial p}{\partial x} = \rho \frac{\partial^2 \xi}{\partial t^2}$$

$$\underset{\sim}{i} k \underset{\sim}{A} e^{i(\omega t - kx)} - \underset{\sim}{i} k \underset{\sim}{B} e^{i(\omega t + kx)} = \rho \frac{\partial^2 \xi}{\partial t^2}$$

$$\frac{\underset{\sim}{i} k \underset{\sim}{A}}{\underset{\sim}{i} \omega \rho} e^{i(\omega t - kx)} - \frac{i k B}{\underset{\sim}{i} \omega \rho} e^{i(\omega t + kx)} = \frac{\partial \xi}{\partial t}$$

$$\frac{\underset{\sim}{A}}{i \omega \rho c} e^{i(\omega t - kx)} - \frac{\underset{\sim}{B}}{i \omega \rho c} e^{i(\omega t + kx)} = \xi(x,t)$$

Since the pressure is not a function of y or z, η and ζ are 0 from (6.4). It turns out that the particle velocity rather than the particle displacement is the more widely used acoustic variable. The x, y, and z-components of the particle velocity are simply $\frac{\partial \xi}{\partial x}$, $\frac{\partial \eta}{\partial y}$ and $\frac{\partial \zeta}{\partial z}$. Letting u = $\partial \xi / \partial x$ be the x-component of particle velocity one has for the pressure waves represented by (6.20)

$$\underset{\sim}{u}(x,t) = \frac{\underset{\sim}{A}}{\rho c} e^{i(\omega t - kx)} - \frac{\underset{\sim}{B}}{\rho c} e^{i(\omega t + kx)} \qquad (6.21)$$

The specific acoustic impedance z at a point in a fluid is defined by

$$z = \frac{p}{u}$$

where p is the acoustic pressure at the point and u is the particle velocity at the point. If the pressure waves represented by (6.20) existed in a fluid then at any point

$$z = \frac{A e^{i(\omega t - kx)} + B e^{i(\omega t + kx)}}{\frac{A}{\rho c} e^{i(\omega t - kx)} - \frac{B}{\rho c} e^{i(\omega t + kx)}} = \rho c \; \frac{e^{-ikx} + \frac{B}{A} e^{ikx}}{e^{-ikx} - \frac{B}{A} e^{ikx}}$$

The specific acoustic impedance is thus a function of x. If $B = 0$ then (6.20) represents a plane progressive wave and the specific acoustic impedance

$$z = \rho c$$

is a constant, the same at all points. This constant impedance ρc is called the <u>characteristic impedance</u> of the medium. The units of specific acoustic impedance are kg sec/m^2 or rayls.

6.6 Transmission and Reflection at a Boundary - Normal Incidence

A progressive plane wave incident on the boundary separating two media, in general, gives rise to a reflected and transmitted wave. After a steady state has been established there will exist in the first medium two waves, the incident and reflected waves. Only a single wave will exist in the second medium assuming it is infinite in extent. For the case of normal incidence illustrated in Fig. 6.5, if

$$p_i = A_1 e^{i(\omega t - k_1 x)} \qquad k_1 = \omega / c_1$$

$$p_r = B_1 e^{i(\omega t + k_1 x)} \qquad k_2 = \omega / c_2$$

$$p_t = A_2 e^{i(\omega t - k_2 x)}$$

represents the incident, reflected and transmitted waves respectively, then at any point to the left of the boundary the acoustic pressure will be given by

$$\underset{\sim}{p}_L = \underset{\sim}{A}_1 e^{i(\omega t - k_1 x)} + \underset{\sim}{B}_2 e^{i(\omega t + k_1 x)}$$

and on the right by

$$\underset{\sim}{p}_R = \underset{\sim}{A}_2 e^{i(\omega t - k_2 x)}$$

The corresponding particle velocity at any point on the left is

$$\underset{\sim}{u}_L = \frac{\underset{\sim}{A}_1}{\rho_1 c_1} e^{i(\omega t - k x)} - \frac{\underset{\sim}{B}}{\rho_1 c_1} e^{i(\omega t + k_1 x)}$$

and on the right

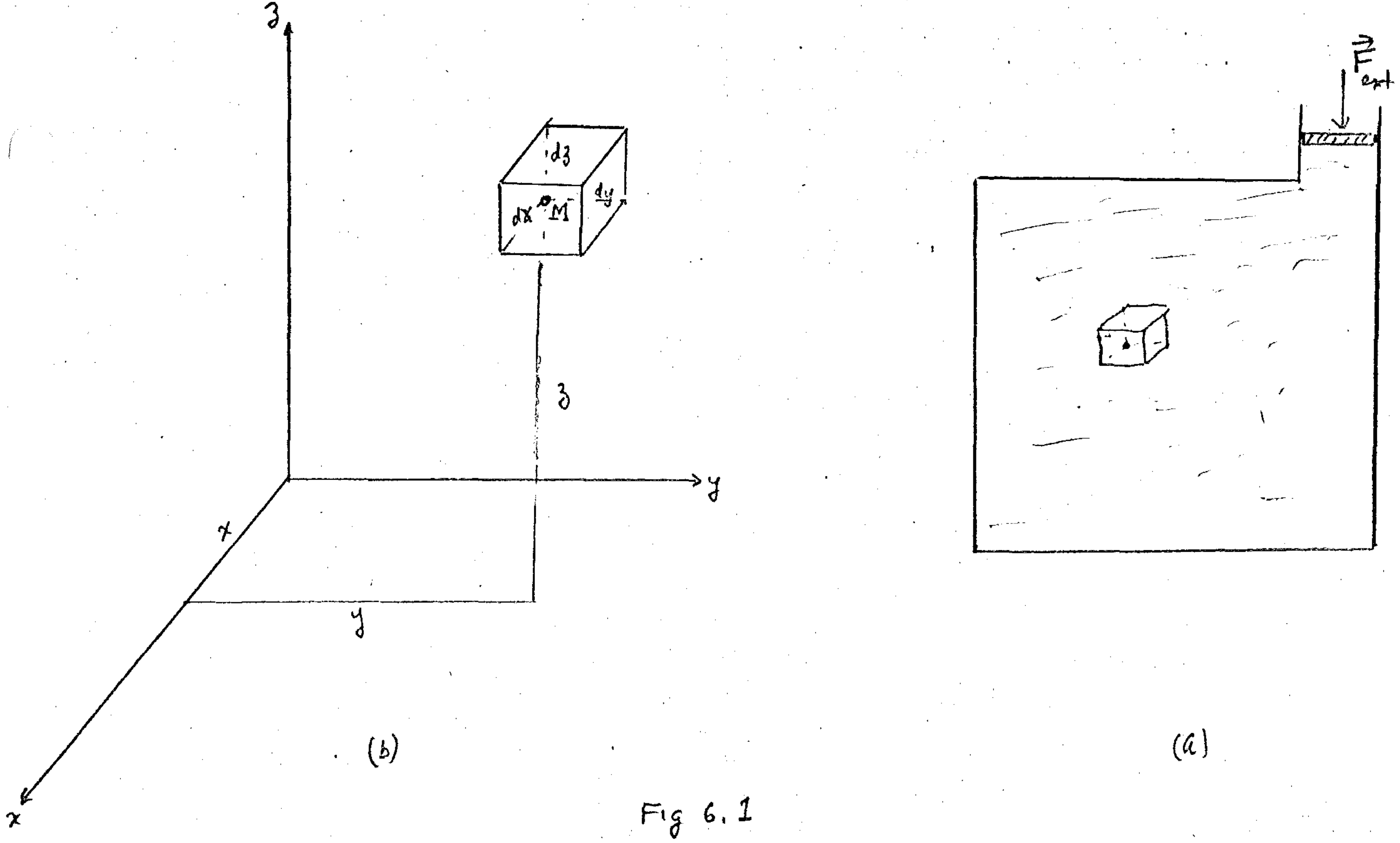
$$\underset{\sim}{u}_R = \frac{\underset{\sim}{A}_2}{\rho_2 c_2} e^{i(\omega t - k_2 x)}$$

At any interface it is generally assumed that the stress (in this instance the pressure) is continuous across the boundary, i.e. that the value of the pressure calculated approaching the boundary from the left must equal the pressure calculated approaching the boundary from the right. It is also assumed that the particle displacement at right angles to the boundary must also be the same approaching the boundary from the left or right. If this were not true, e.g. if the two particles labelled ① and ② in Fig. 6.5 did not move simultaneously to the right or left, a gap would appear in the boundary. If the particle displacement at right angles to the boundary is continuous, it follows that the component of particle velocity at right angles to the boundary is also continuous. Letting the interface be located at the origin of the coordinate system for convenience, the boundary conditions for the case illustrated in Fig. 6.5 yields

$$\underset{\sim}{A}_1 + \underset{\sim}{B}_1 = \underset{\sim}{A}_2$$

$$\frac{\underset{\sim}{A}_1}{\rho_1 c_1} - \frac{\underset{\sim}{B}_1}{\rho_1 c_1} = \frac{\underset{\sim}{A}_2}{\rho_2 c_2}$$

from which one obtains



(b) (a)

Fig 6.1

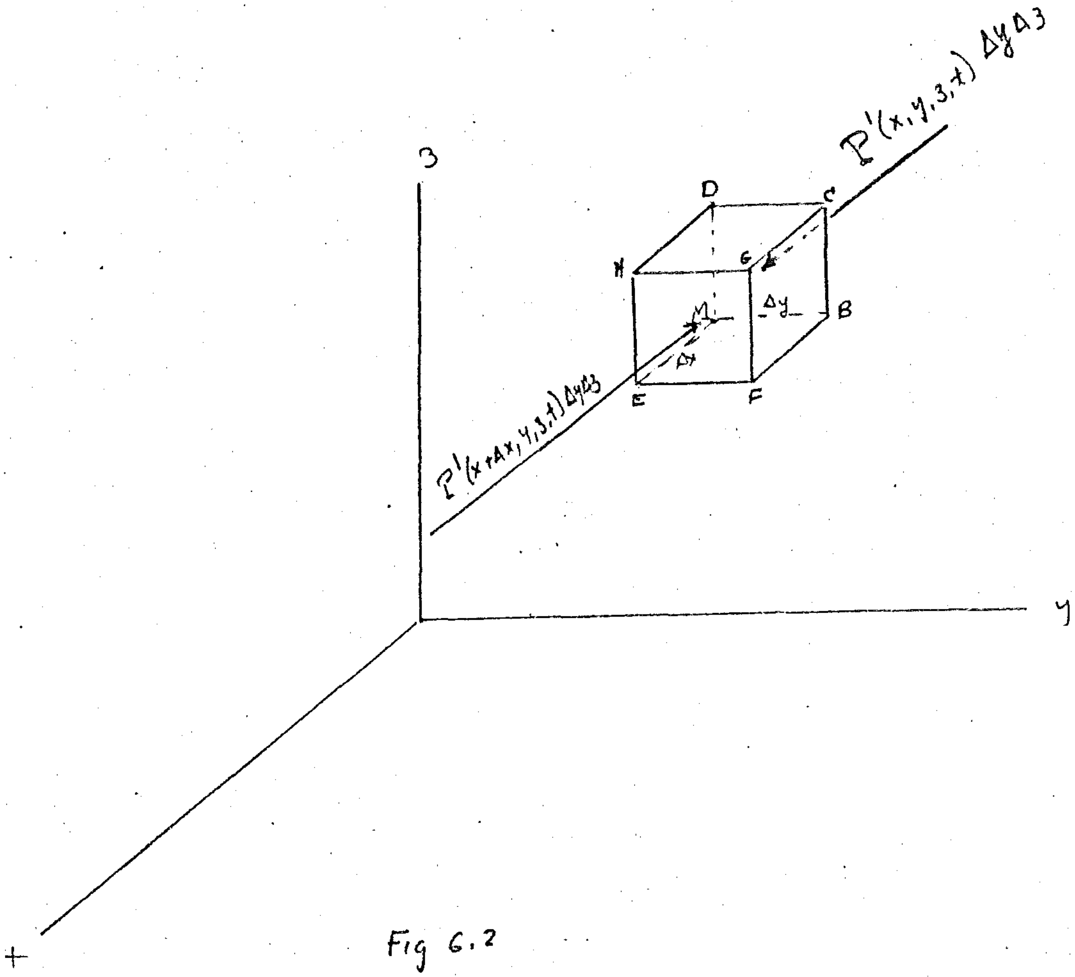


Fig 6.2

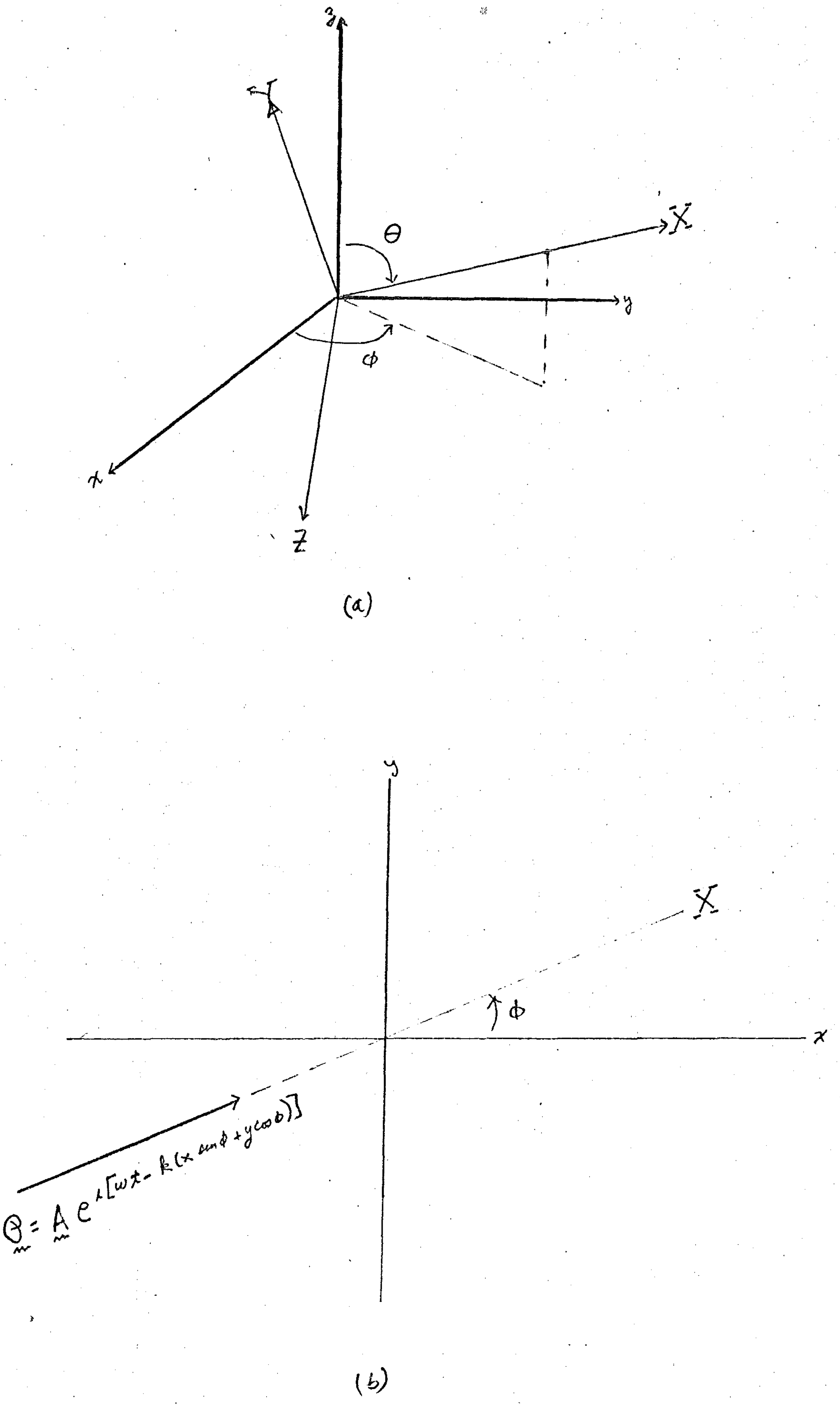

z
Y
A
θ
X
y
φ
x
Z
(a)
y
X
φ
x
$\underset{\sim}{\Theta} = \underset{\sim}{A}\, e^{i[\omega t - k(x \sin\phi + y \cos\phi)]}$
(b)

Fig 6,3

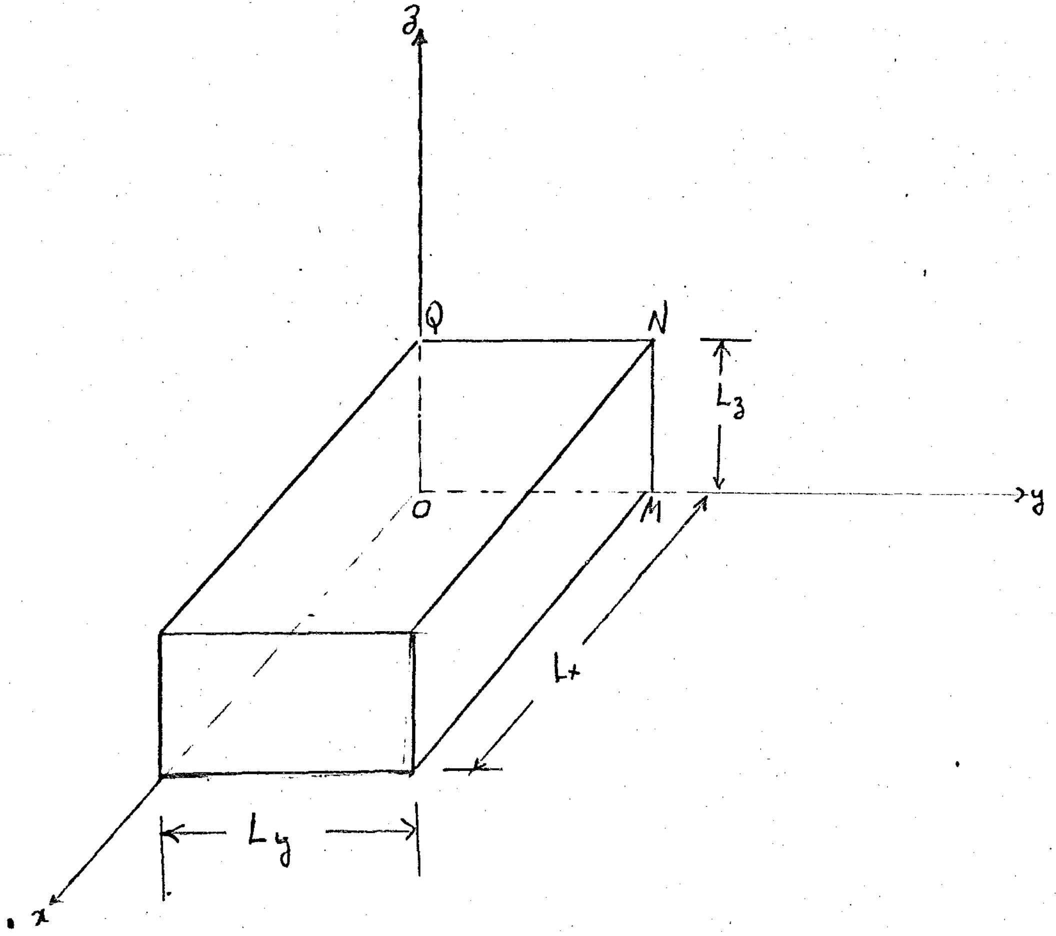
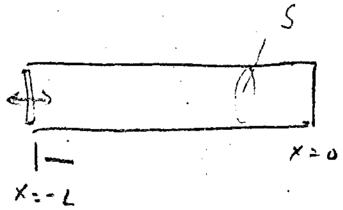


Fig 6.4

$$\dot{x}_{P_{real}} = \frac{F_0/\rho c S}{\sqrt{R_1^2(2ka) + \left[\bar{X}_1(2ka) - \cot kL\right]^2}} \cos(\omega t + \beta)$$



$$\underset{\sim}{P} = \underset{\sim}{A} \cos kx \, e^{i\omega t}$$

$$\underset{\sim}{u} = -\frac{i}{\rho c} \underset{\sim}{A} \sin kx \, e^{i\omega t}$$

$$\underset{\sim}{\dot{x}}_0 = \frac{F_0 e^{i\omega t} - \underset{\sim}{P}\big|_{x=-L} S}{\underset{\sim}{Z}_m + \underset{\sim}{Z}_r} = \underset{\sim}{u}\Big|_{x=-L}$$

$$\frac{F_0 e^{i\omega t} - \underset{\sim}{A} e^{i\omega t} S \cos kL}{\underset{\sim}{Z}_m + \underset{\sim}{Z}_r} = +\frac{i}{\rho c} \underset{\sim}{A} \sin kL \, e^{i\omega t}$$

$$F_0 - \underset{\sim}{A} S \cos kL = \left(\underset{\sim}{Z}_m + \underset{\sim}{Z}_r\right) \frac{i}{\rho c} \underset{\sim}{A} \sin kL$$

$$F_0 = \left[\left(\underset{\sim}{Z}_m + \underset{\sim}{Z}_r\right) \frac{i}{\rho c} \sin kL + S \cos kL\right] \underset{\sim}{A}$$

$$\underset{\sim}{A} = \frac{F_0}{\left(\underset{\sim}{Z}_m + \underset{\sim}{Z}_r\right) \frac{i}{\rho c} \sin kL + S \cos kL}$$

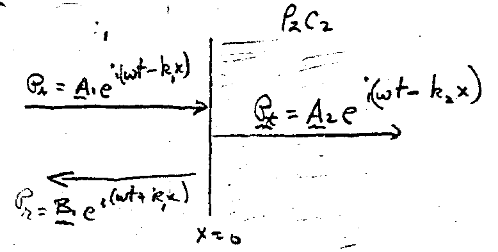
$$\underset{\sim}{P}\Big|_{x=0} = \frac{F_0 e^{i\omega t}}{\left(\underset{\sim}{Z}_m + \underset{\sim}{Z}_r\right) \frac{i}{\rho c} \sin kL + S \cos kL}$$

Neglecting $\underset{\sim}{Z}_m$ in comparison to $\underset{\sim}{Z}_r$

$$\underset{\sim}{P}\Big|_{x=0} = \frac{F_0 e^{i\omega t}}{\rho c S\left[R_1 + iX_1\right] \frac{i}{\rho c} \sin kL + S \cos kL} = \frac{F_0 e^{i\omega t}}{S\left\{\left[\cos kL - \bar{X}_1 \sin kL\right] + i R_1 \sin kL\right\}}$$

$$P_{real}\Big|_{x=0} = \frac{F_0}{S\sqrt{\left(\cos kL - X_1 \sin kL\right)^2 + R_1^2 \sin^2 kL}} \cos(\omega t - \alpha)$$

Two media



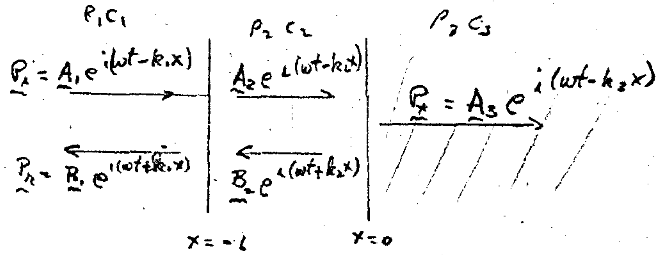
$$\underset{\sim}{P}_L\Big|_{x=0} = \underset{\sim}{P}_R\Big|_{x=0} \Rightarrow \underset{\sim}{A}_1 + \underset{\sim}{B}_1 = \underset{\sim}{A}_2$$

$$\underset{\sim}{u}_L\Big|_{x=0} = \underset{\sim}{u}_R\Big|_{x=0} \Rightarrow \frac{\underset{\sim}{A}_1}{\rho_1c_1} - \frac{\underset{\sim}{B}_1}{\rho_1c_1} = \frac{\underset{\sim}{A}_2}{\rho_2c_2}$$

$$\Rightarrow \frac{\underset{\sim}{A}_1+\underset{\sim}{B}_1}{\underset{\sim}{A}_1-\underset{\sim}{B}_1} = \frac{\rho_2c_2}{\rho_1c_1}$$

$$\frac{\underset{\sim}{B}_1}{\underset{\sim}{A}_1} = \frac{\rho_2c_2/\rho_1c_1 - 1}{\frac{\rho_2c_2}{\rho_1c_1}+1}$$

Three media



$$\underset{\sim}{P}_L\Big|_{x=0} = \underset{\sim}{P}_R\Big|_{x=0} \Rightarrow \underset{\sim}{A}_2 + \underset{\sim}{B}_2 = \underset{\sim}{A}_3$$

$$\underset{\sim}{u}_L\Big|_{x=0} = \underset{\sim}{u}_R\Big|_{x=0} \Rightarrow \frac{\underset{\sim}{A}_2}{\rho_2c_2} + \frac{\underset{\sim}{B}_2}{\rho_2c_2} = \frac{\underset{\sim}{A}_3}{\rho_3c_3}$$

$$\Rightarrow \frac{\underset{\sim}{B}_2}{\underset{\sim}{A}_2} = \frac{\frac{\rho_3c_3}{\rho_2c_2}-1}{\frac{\rho_2c_2}{\rho_1c_1}+1} = \frac{r_{23}-1}{r_{23}+1}$$

$$\underset{\sim}{P}_L\Big|_{x=-L} = \underset{\sim}{P}_R\Big|_{x=-L} \Rightarrow \underset{\sim}{A}_1 e^{\cancel{ik_1L}} + \underset{\sim}{B}_1 e^{-ik_1L} = \underset{\sim}{A}_2 e^{ik_2L} + \underset{\sim}{B}_2 e^{-ik_2L}$$

$$\underset{\sim}{u}_L\Big|_{x=-L} = \underset{\sim}{u}_R\Big|_{x=-L} \Rightarrow \frac{\underset{\sim}{A}_1}{\rho_1c_1} e^{+ik_1L} - \frac{\underset{\sim}{B}_1}{\rho_1c_1} e^{-ik_1L} = \frac{\underset{\sim}{A}_2}{\rho_2c_2} e^{ik_2L} - \frac{\underset{\sim}{B}_2}{\rho_2c_2} e^{-ik_2L}$$

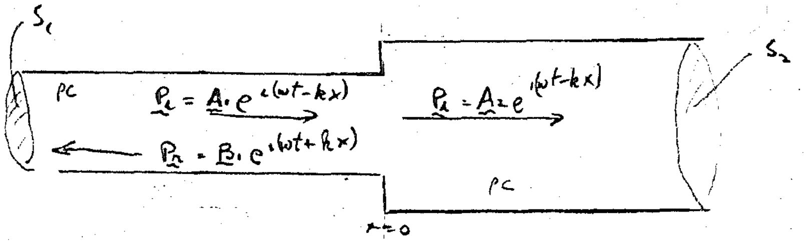
$$\Downarrow$$

$$\underset{\sim}{B}_1 = \frac{(r_{13}-1)\cos k_2L + i(r_{12}-r_{23})\sin k_2L}{(r_{13}+1)\cos k_2L + i(r_{12}+r_{23})\sin k_2L}\left[\underset{\sim}{A}_1 e^{2ik_1L}\right]$$

$$r_{13} = \frac{\rho_3c_3}{\rho_1c_1}$$

$$r_{23} = \frac{\rho_3c_3}{\rho_2c_2} \qquad r_{12} = \frac{\rho_2c_2}{\rho_1c_1}$$

ONE JUNCTION



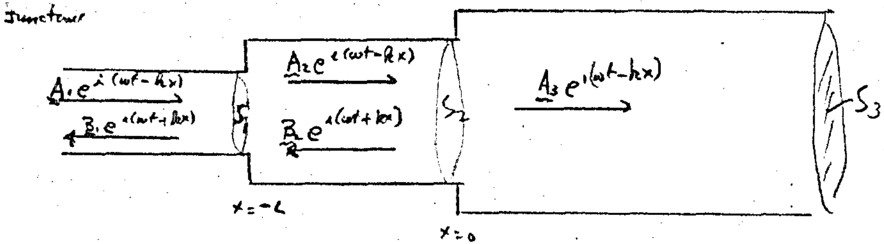
$$\underset{\sim}{P}_L\Big|_{x=0} = \underset{\sim}{P}_R\Big|_{x=0} \Rightarrow \underset{\sim}{A}_1 + \underset{\sim}{B}_1 = \underset{\sim}{A}_2$$

$$\underset{\sim}{u}_L S_1\Big|_{x=0} = \underset{\sim}{u}_R S_2\Big|_{x=0} \Rightarrow \left[\frac{\underset{\sim}{A}_1}{\rho c} - \frac{\underset{\sim}{B}_1}{\rho c}\right] S_1 = \frac{\underset{\sim}{A}_2}{\rho c} S_2$$

$$\Rightarrow \frac{\underset{\sim}{A}_1+\underset{\sim}{B}_1}{\underset{\sim}{A}_1-\underset{\sim}{B}_1} = \frac{S_1}{S_2}$$

$$\frac{\underset{\sim}{B}_1}{\underset{\sim}{A}_1} = \frac{S_1/S_2 - 1}{S_1/S_2 + 1}$$

Two Junctions



$$\underset{\sim}{P}_L\Big|_{x=0} = \underset{\sim}{P}_R\Big|_{x=0} \Rightarrow \underset{\sim}{A}_2 + \underset{\sim}{B}_2 = \underset{\sim}{A}_3$$

$$\underset{\sim}{u}_L S_2\Big|_{x=0} = \underset{\sim}{u}_R S_3\Big|_{x=0} \Rightarrow \left[\frac{\underset{\sim}{A}_2}{\rho c} - \frac{\underset{\sim}{B}_2}{\rho c}\right] S_2 = \frac{\underset{\sim}{A}_3}{\rho c} S_3$$

$$\Rightarrow \frac{\underset{\sim}{B}_2}{\underset{\sim}{A}_2} = \frac{\frac{S_2}{S_3}-1}{\frac{S_2}{S_3}-1} = \frac{r_{32}-1}{r_{32}+1}$$

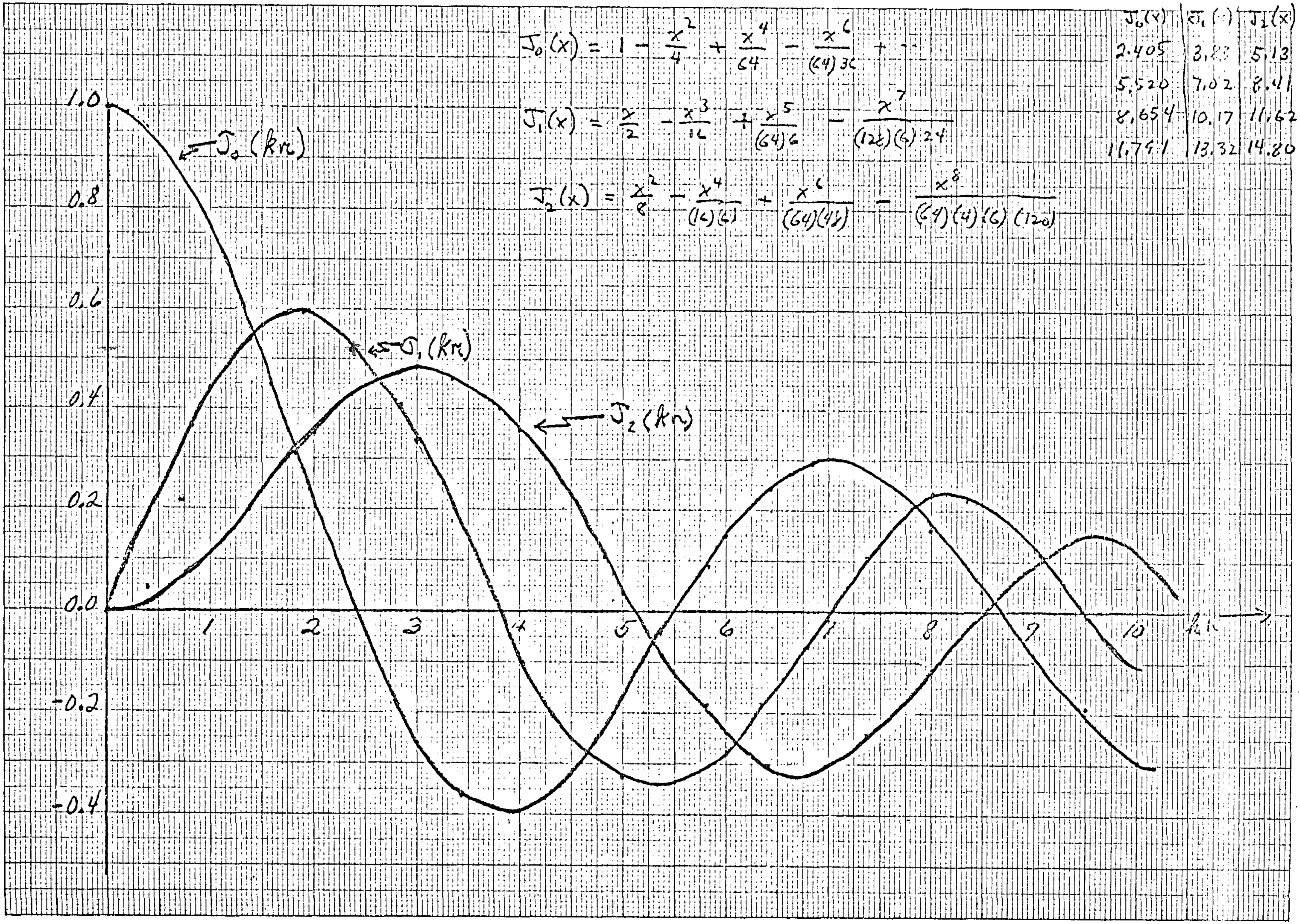
$$\underset{\sim}{P}_L\Big|_{x=-L} = \underset{\sim}{P}_R\Big|_{x=-L} \Rightarrow \underset{\sim}{A}_1 e^{+ikL} + \underset{\sim}{B}_1 e^{-ikL} = \underset{\sim}{A}_2 e^{ikL} - \underset{\sim}{B}_2 e^{-ikL}$$

$$u_L S_1\Big|_{x=-L} = u_R S_2\Big|_{x=-L} \Rightarrow \left[\frac{\underset{\sim}{A}_1}{\rho c} e^{ikL} - \frac{\underset{\sim}{B}_1}{\rho c} e^{-ikL}\right] S_1 = \left[\frac{\underset{\sim}{A}_2}{\rho c} e^{ikL} - \frac{\underset{\sim}{B}_2}{\rho c} e^{-ikL}\right] S_2$$

$$\Downarrow$$

$$\underset{\sim}{B}_1 = \frac{(r_{31}-1)\cos kL + i(r_{21}-r_{32})\sin kL}{(r_{31}+1)\cos kL + i(r_{21}+r_{32})\sin kL}\left[\underset{\sim}{A}_1 e^{2ikL}\right]$$

$$r_{31} = \frac{S_1}{S_3} \; ; \; r_{21} = \frac{S_1}{S_2} \qquad r_{32} = \frac{S_2}{S_3}$$



$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{(64)36} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{(64)6} - \frac{x^7}{(128)(6)24}$$

$$J_2(x) = \frac{x^2}{8} - \frac{x^4}{(16)(6)} + \frac{x^6}{(64)(48)} - \frac{x^8}{(64)(4)(6)(120)}$$

Zeros

$J_0(x)$	$J_1(x)$	$J_2(x)$
2.405	3.83	5.13
5.520	7.02	8.41
8.654	10.17	11.62
11.791	13.32	14.80

Fig. 4-7

TABLE 5.1 BESSEL FUNCTIONS (FIRST KIND)

$$J_n = \frac{x^n}{2^n n!}\left\{1 - \frac{x^2}{2\cdot(2n+2)} + \frac{x^4}{2\cdot 4\cdot(2n+2)(2n+4)} - \frac{x^6}{2\cdot 4\cdot 6\cdot(2n+2)(2n+4)(2n+6)} + \cdots\right\} \qquad n=0,1,2,3\ldots$$

$$\frac{d}{dx}\left[J_0(x)\right] = -J_1(x)$$

$$J_0(x) = 1 - \int_0^x J_1(x')\,dx'$$

$$\left.\begin{array}{l} \dfrac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \\[2ex] \dfrac{d}{dx}\left[J_n(x)\right] = \dfrac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \\[2ex] J_n(x) = \dfrac{1}{2n}\left[x\, J_{n-1}(x) + x\, J_{n+1}(x)\right] \end{array}\right\} \quad n=1,2,3\ldots$$

When x is large ($x > 10$), approximate values of $J_n(x)$ may be computed from the semi-convergent series

$$J_n(x) = \sqrt{\frac{2}{\pi x}}\left[P_n \cos\left\{\frac{(2n+1)\pi}{4} - x\right\} + Q_n \sin\left\{\frac{(2n+1)\pi}{4} - x\right\}\right]$$

where

$$P_n = 1 - \frac{(4n^2-1)(4n^2-9)}{2!(8x)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)(4n^2-49)}{4!(8x)^4} - \cdots$$

$$Q_n = \frac{4n^2-1}{8x} - \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!\,(8x)^3} + \cdots$$

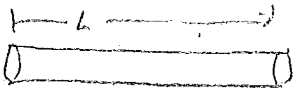
Arguments for which Bessel Functions are Zero

J_0	J_1	J_2	J_3	J_4	J_5
2,4048	0.	0.	0,	0	0
5.5201	3,8317	5,1356	6,3802	7,5883	8,7715
8.6537	7,0156	8,4172	9.7610	11,0647	12,3386
11,7915	10.1735	11,6198	13.0152	14,3725	15,7002
14,9309	13,3237	14.7960	16,2235	17,6160	18,9801
	16.4706	17.9598	19.4094	20.8269	22,2178

$L = 52.7\,cm$
$= .0527\ M$

$a = .0182\ m$

$C = 343\ m/sec$



f	$ka = \frac{2\pi f}{c} a$	$kL = \frac{2\pi f}{c} L$	$R_1(2ka)$	$X_1(2ka)$	$\cot kL$
20	.006	.193	<.005	<.085	5.14
40		.387			2.45
60		.580			1.54
80		.774			1.02
100	.033	.966			.691
120		1.16			.434
140		1.35			.222
160		1.54			.032
180		1.74			-.176
200	.067	1.93			-.364

find the phase difference between $x_1(t)$ and $x_2(t)$ and the ratio of the amplitude of $x_1(t)$ to that of $x_2(t)$.

2.7 If

$$\underset{\sim}{P_i} = \underset{\sim}{A_1} e^{i\omega t} \quad ; \quad \underset{\sim}{P_r} = \underset{\sim}{B_1} e^{i\omega t} \quad ; \quad \underset{\sim}{P_t} = \underset{\sim}{A_2} e^{i\omega t}$$

and

$$\underset{\sim}{P_i} + \underset{\sim}{P_r} = \underset{\sim}{P_t}$$

$$\underset{\sim}{P_i} - \underset{\sim}{P_r} = \lambda P_t$$

find the phase difference between $P_i(t)$ and $P_r(t)$ and the ratio of the amplitude of $P_r(t)$ and that of $P_i(t)$.

2.8 The solution of the dampled harmonic oscillator has the form

$$x(t) = A e^{-\alpha t} \cos(\omega_D t + \phi)$$

This function of t' has a series of maxima and minima. The condition that $x(t)$ have a maximum or minimum is that $\frac{dx}{dt} = 0$, i.e. the maxima and minima occur at those times when the velocity is zero. Show that the velocity is zero at times t which satisfy the condition

$$\tan(\omega_D t + \phi) = -\frac{\alpha}{\omega_D} \implies t = \frac{\tan^{-1}\left(-\frac{\alpha}{\omega_D}\right) - \phi}{\omega_D} \qquad (i)$$

If Ψ_1 is the smallest positive angle whose tangent is $(-\alpha/\omega_D)$ then every angle

$$\Psi_n = \Psi_1 + n\pi \qquad n = 0, 1, 2, 3 \ldots$$

will also have a tangent equal to $(-\alpha/\omega_0)$. Thus the values of time t_n for which (i) is satisfied are

$$t_n = \frac{\psi_1 + n\pi - \phi}{\omega_b} \qquad n = 0, 1, 2, 3 \ldots$$

Show that the ratio of two successive maxima (or two successive minima) of $x(t)$ is constant and equal to $e^{\frac{2\pi\alpha}{\omega_b}}$.

2.9 Show that it is possible to express the coefficients a_3, a_4, a_5 .... in terms of a_0 and a_1 so that the series

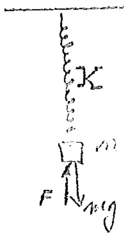
$$x(t) = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots$$

will be a solution of

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0$$

the equation of motion of the damped harmonic oscillator.

2.10 A mass m on the end of a spring of force constant K is held in equilibrium by a force F_0, equal in magnitude to the gravitational force mg. Find the subsequent motion of the mass if the force F_0 is suddenly removed. Assume a damping force proportional to velocity and neglect the mass of the spring.



2.11 A simple harmonic oscillator of mass m, spring constant K is set in motion by a sharp blow. Assume the impulse of the blow is I_0. Find the subsequent motion of the oscillator assuming a damping force proportional to the velocity.

2.12 A certain damped harmonic oscillator is found to have a period τ_0 of 1/2 sec and an $\alpha = -R/2m$ of 0.1 $\sec^{-1}$. If this oscillator were driven by a force $F_0 \cos \omega t$, at what frequency ω would resonance occur?

2.13 A drvving force $F_0 \cos \omega t$ is applied to damped harmonic oscillator at a time $t = 0$ when the oscillator is at rest in its equilibrium position. Describe the subsequent motion of the oscillator.

2.14 Show that

$$\frac{1}{\tau}\int_0^{\tau} \cos^2(\omega t + kx + \alpha)\, dt = \frac{1}{\tau}\int_0^{\tau} \sin^2(\omega t + kx + \alpha)\, dt = \frac{1}{2}$$

and

$$\frac{1}{\tau}\int_0^{\tau} \cos(\omega t + kx) \cos(\omega t + kx + \theta)\, dt = \cos\theta/2$$

where $\omega = 2\pi/\tau$ and k, α, x and Θ are arbitrary constants.

2.15 In the steady state the motion of an harmonic oscillator driven by a force $F_0 \cos \omega t$ is given by

$$x = \frac{F_0/\omega}{Z_m} \sin(\omega t - \theta) \qquad \text{where} \qquad Z_m = \sqrt{R^2 + (\omega m - K/\omega)}$$

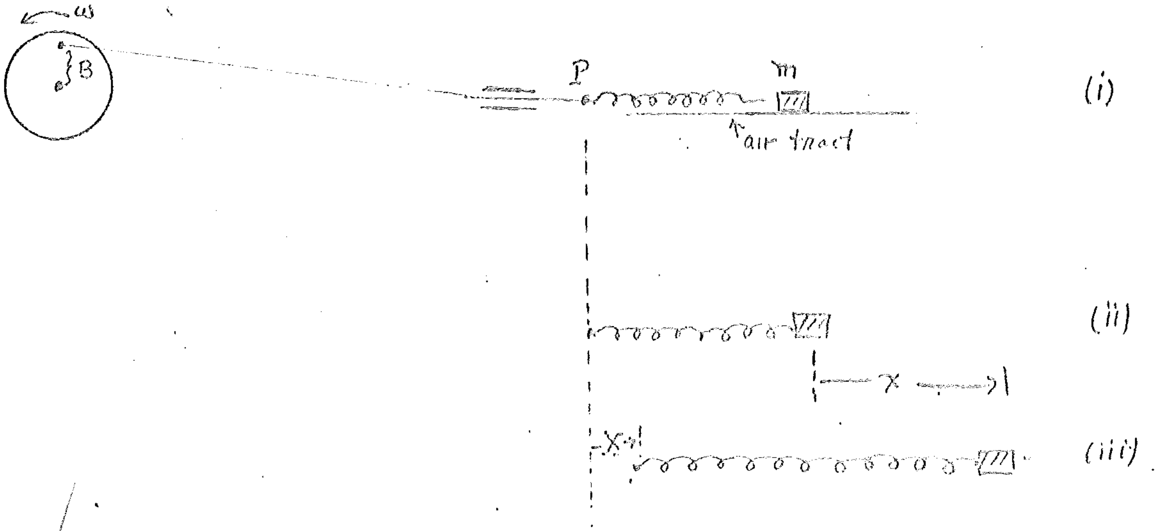
$$\dot{x} = \frac{F_0}{Z_m} \cos(\omega t - \theta) \qquad\qquad \tan\theta = \frac{\omega m - K/\omega}{R}$$

For obvious reasons the quantity $F_o/\omega Z_m$ is referred to as the displacement amplitude, while the quantity F_o/Z_m is referred to as the velocity amplitude. If the angular frequency ω of the driving force is varied keeping F_o constant, and for each frequency the displacement and velocity amplitudes are noted, find in terms of m, K and R, the angular frequency at which the displacement amplitude would be largest. Find the frequency at which the velocity amplitude would be largest.

(2.16) It is possible to apply a force of the form $F_o \cos \omega t$ to an harmonic oscillator by means of the arrangement shown in the figure (i). The end P of the spring is fastened by a light rod [redacted] to a peg on a wheel mounted on the shaft of a motor which rotates with an adustable angular velocity ω . [redacted] point P is forced to move (very nearly) with simple harmonic motion, so that its motion is given by $x = B \cos \omega t$. Fig (ii) shows the system at some instant when the spring is <u>unstretched</u> and point P is at the midpoint of its motion. Fig (iii) shows the system at some general time t. Isolate the mass m in this last figure, draw in the force exerted by the spring and assume an additional damping force $R\dot{x}$. Write down the equation of motion and show that this has the form

$$m\ddot{x} + R\dot{x} + Kx = F_o \cos \omega t$$

How is F_0 related to B. Let A_1 be the <u>displacement</u> amplitude of the system when the system is at resonance, i.e. when $\omega = \sqrt{K/m}$ Show that the Q of this system is equal to A_1/B.



2.17 An harmonic oscillator is being driven by a driving force $F_0 \cos \omega t$ at such a frequency that

$$\omega m = 3 \text{ kg/sec}$$

$$K/\omega = 5 \text{ kg/sec}$$

$$R = 2 \text{ kg/sec}$$

Is the driving frequency smaller than, equal to , or greater than the resonant frequency? What is the phase difference between the driving force and the displacement x? Which leads? What is the mechanical impedance $\underset{\sim}{Z_m}$ of the oscillator at this frequency? What is the Q of this mechanical system?

2.18 In the steady state, is the rate at which the driving force supplies energy to a damped harmonic oscillator equal at every instant to the rate energy is being dissipated? Is the total mechanical energy (potential plus kinetic) of a driven damped oscillator a constant in the steady state?

BOB MARKS

$\frac{9.4}{10}$

2-2) [illegible]

a) $x(t) = 4\,($ [illegible] $)$

$Re\{x(t)\} = 4\cos\pi t$ ✓

b) $\omega_0 = \pi \frac{RAD}{SEC}$

$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2}\ \frac{CYCLE}{SEC}$ ✓

c) $|x(t)| = 4\,|e^{i\pi t}|$

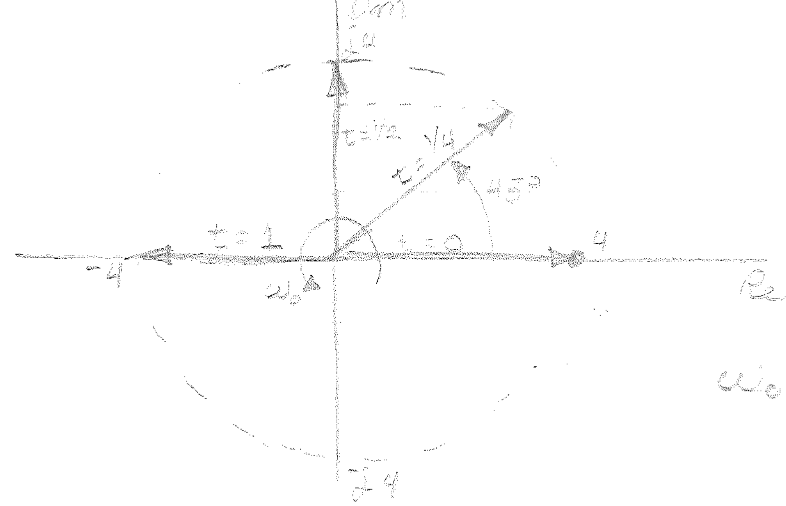
$= 4$ ✓

d) $x(0) = 4$

$x(1/4) = 4e^{\frac{i\pi}{4}}$

$x(1/2) = 4e^{\frac{i\pi}{2}}$

$x(1) = 4e^{i\pi}$



$\omega_0 = \pi\ \frac{RAD}{SEC}$ ✓

2-4) $\underset{\sim}{x}(t) = 4e^{i\pi t} = 4(\cos\pi t + j\sin\pi t)$

$Re\{\underset{\sim}{x}(t)\} = 4\cos\pi t \Rightarrow \omega = \pi$

$\underset{\sim}{x}_1(t) = (3 + i4)e^{i\pi t}$

$= (3 + i4)(\cos\pi t + i\sin\pi t)$

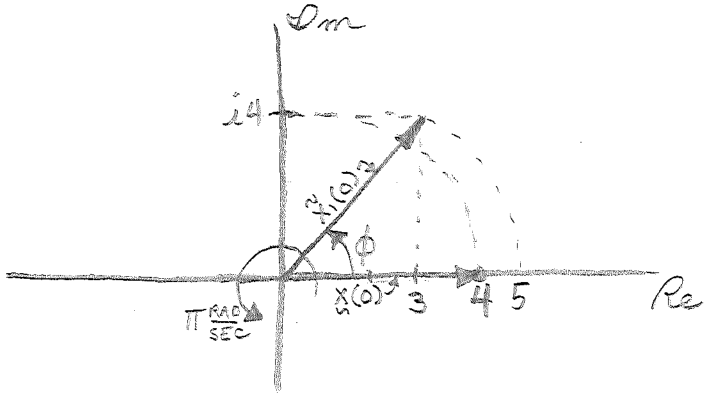
$= (3\cos\pi t - 4\sin\pi t) + i(4\cos\pi t + 3\sin\pi t)$

$Re\{\tilde{x}_1(t)\} = 3\cos\pi t - 4\sin\pi t \Rightarrow \omega_1 = \pi$

⇒ BOTH HAVE THE SAME <u>FREQUENCY</u> ✓ $(\omega_0 = \pi)$

$|\underset{\sim}{x}(t)| = 4 \quad ; \quad |\underset{\sim}{x}_1(t)| = \sqrt{3^2 + 4^2} = 5$

⇒ BOTH DO NOT HAVE THE SAME AMPLITUDE ✓



PHASE DIFFERENCE $= \phi = atan\left(\frac{4}{3}\right) = 51.4°$ ✓

$\tilde{x}_1(t)$ LEADS $\underset{\sim}{x}(t)$ BY 51.4° ✓

2-7) $P_i = A_1 e^{i\omega t}$ $\quad P_r = B_1 e^{i\omega t}$ $\quad P_* = A_2 e^{i\omega t}$

$$P_i + P_r = P_* \Rightarrow A_1 e^{i\omega t} + B_1 e^{i\omega t} = A_2 e^{i\omega t}$$

$$A_1 + B_1 = A_2$$

$$P_i - P_r = 2P_* \Rightarrow A_1 e^{i\omega t} - B_1 e^{i\omega t} = 2A_2 e^{i\omega t}$$

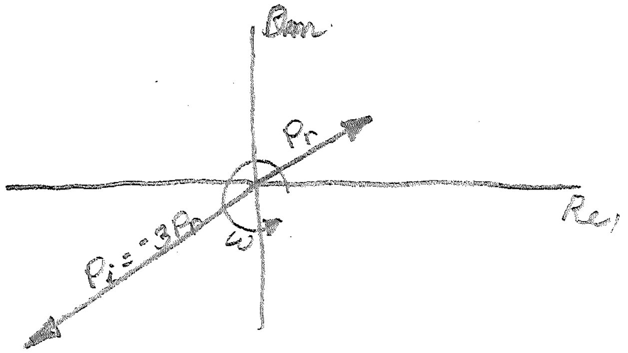
$$A_1 - B = 2A_2$$

$$2A_1 + 2B_1 = 2A_2$$

$\therefore \quad A_1 + 3B_1 = 0 \Rightarrow A_1 = {}^{-}3B_1$

THUS $\quad P_i = {}^{-}3B_1 e^{i\omega t} \;;\; P_r = B_1 e^{i\omega t}$

OR $\quad P_i = {}^{-}3P_r$



ERGO, P_i & P_r ARE 180° OUT OF PHASE, i.e. P_i & P_r HAVE A PHASE DIFFERENCE OF 180°, EITHER LEADING OR LAGGING AND $\left|\frac{P_i}{P_r}\right| = 3$

2.9) $\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0$

LET $x(t) = \sum_{n=0}^{\infty} a_n t^n$

THEN $\dot{x} = \sum_{n=1}^{\infty} n a_n t^{n-1}$

AND $\ddot{x} = \sum_{n=2}^{\infty} (n)(n-1) a_n t^{n-2}$

$$\Rightarrow \omega_0^2 x = \omega_0^2[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \ldots]$$

$$2\alpha\dot{x} = 2\alpha[a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5 + \ldots]$$

$$\ddot{x} = [2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 30a_6 t^4 \ldots]$$

ADDING THE ABOVE EQUATIONS, GROUPING LIKE TERMS:

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0$$

$$= (2a_2 + 2\alpha a_1 + \omega_0^2 a_0) + (6a_3 + 4\alpha a_2 + \omega_0^2 a_1)t$$

$$+ (12a_4 + 6\alpha a_3 + \omega_0^2 a_2)t^2 + (20a_5 + 8\alpha a_4 + \omega_0^2 a_3)t^3$$

$$+ (30a_6 + 10\alpha a_5 + \omega_0^2 a_4)t^4 + \ldots$$

$$= \sum_{n=2}^{\infty}[n(n-1)a_n + 2n\alpha a_{n-1} + \omega_0^2 a_{n-2}]t^{n-2}$$

LET $m = n-2 \Rightarrow n = m+2$

$$\Rightarrow \ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = 0 = \sum_{m=0}^{\infty}[(m+2)(m+1)a_{m+2} + 2(m+2)\alpha a_{m+1} + \omega_0^2 a_m]t^m$$

THIS IDENITY WILL HOLD IF t^m COEFFICIENTS ARE ALL IDENTICALLY ZERO FOR ALL m

$$\Rightarrow (m+2)(m+1)a_{m+2} + 2(m+2)\alpha a_{m+1} + \omega_0^2 a_m = 0$$

OR $a_{m+2} = \dfrac{2(m+2)\alpha a_{m+1} + \omega_0^2 a_m}{-(m+2)(m+1)} = f(a_{m+1}, a_m)$

THEN $a_2 = f(a_0, a_1)$

$a_3 = f(a_2, a_1) = f[f(a_0, a_1), a_1]$

$a_4 = f(a_3, a_2) = f[f\{f(a_0, a_1), f(a_0, a_1)]$

EXTENSION WILL YIELD THE VALUE OF ANY a_n $(n \geq 2)$ AS FUNCTIONS OF ONLY a_0 AND a_1 ✓

2-11) $m\ddot{x} + kx = I_0\delta(t) \ni \delta(t)$ IS IMPULSE @ ORIGIN

EMPLOYING THE LAPLACE TRANSFORM $\left(I_0 = \int_{0}^{+0} F dt \Rightarrow E = I_0 \ldots\right.$

$$m s^2 X(s) - s \cancel{\frac{dx}{dt}\Big|_{t=0}}^{0} - \cancel{x(0)}^{0} + kX(s) = I_0$$

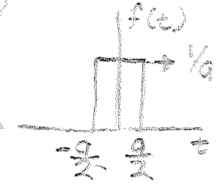
$$\Rightarrow X(s) = \frac{I_0}{ms^2 + k} = \frac{I_0/m}{s^2 + k/m}$$

TAKING INVERSE XFORM:

$$x(t) = \frac{I_0}{m}\sqrt{\frac{m}{K}}\, \sin\sqrt{\frac{K}{m}}\, t$$

$$= \frac{I_0}{m\omega_D}\sin\omega_D t \ni \omega_D = \sqrt{K/m - \alpha^2}$$

(



$;\ \delta(t) = \lim_{a \to 0} f(t)$)

2-12) $m\ddot{x} + R\dot{x} + kx = F_0 \cos \omega t$

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} \quad ; \quad \alpha = \frac{R}{2m}$$

POWER CONSIDERATIONS:

$$P_{AVE} = \frac{F_0 R}{2[R^2 + (\omega m - k/\omega)^2]}$$

$$= \frac{F_0 R}{2m^2[(R/m)^2 + (\omega - k/m\omega)^2]}$$

$$= \frac{F_0 \alpha}{m[4\alpha^2 + (\omega - \omega_0^2/\omega)^2]}$$

RESONANCE OCCURS WHEN P_{AVE} IS MAXIMUM. AS A FUNCTION OF FREQUENCY, THIS OCCURS WHEN $(\omega - \omega_0^2/\omega)^2$ IS MINIMUM. THIS OCCURS WHEN $\omega = \omega_0$, AT WHICH INSTANCE $(\omega - \frac{\omega_0^2}{\omega})^2 = 0$.

THUS, RESONANCE OCCURS @ $\omega = \omega_0 = \frac{2\pi}{T_0}$

$$\omega = \frac{2\pi}{T_0} = \frac{2\pi}{1/2}$$

$\Rightarrow f = 2$ Hz

$$T_D = \frac{1}{2} = \frac{2\pi}{\omega_D} \qquad \omega_D = \sqrt{\frac{k}{m} - \alpha^2} \Rightarrow \sqrt{\frac{k}{m}} = \sqrt{\omega_D^2 + \alpha^2}$$

$$\omega_{m} = \sqrt{\frac{k}{m}} = \sqrt{\omega_D^2 + \alpha^2} \cong \omega_D \text{ if } \alpha << \omega_D$$

2-14) a) $\frac{1}{\tau}\int_0^\tau \cos^2(\omega t+kx+\alpha)\,dt = \frac{1}{2\tau}\int_0^\tau\left[1+\cos 2\{\omega t+kx+\alpha\}\right]dt$

$= \frac{1}{2\tau}\left[t+\frac{1}{2\omega}\sin 2\{\omega t+kx+\alpha\}\right]_0^\tau$

$= \frac{1}{2\tau}\left[\tau+\frac{1}{2\omega}(\sin 2\{\omega\tau+kx+\alpha\}\right.$

$\left.-\frac{1}{2\omega}\sin 2\{kx+\alpha\}\right]$

BUT $\omega=\frac{2\pi}{\tau}$, AND $\sin(At+\phi)=\sin\left[(A+2\pi)t+\phi\right]$

$\Rightarrow \frac{1}{\tau}\int_0^\tau \cos^2(\omega t+kx+\alpha)\,dt = \frac{1}{2\tau}\left[\tau+0\right]=\frac{1}{2}$ ✓

b) $\frac{1}{\tau}\int_0^\tau \sin^2(\omega t+kx+\alpha)\,dt = \frac{1}{2\tau}\int_0^\tau\left[1-\cos 2\{\omega t+kx+\alpha\}\right]dt$

$= \frac{1}{2\tau}\left[t-\frac{1}{2\omega}\sin\{\omega t+kx+\alpha\}\right]_0^\tau$

$= \frac{1}{2\tau}\left[\tau-0\right]=\frac{1}{2}$ ✓

c) $\frac{1}{\tau}\int_0^\tau \cos(\omega t+kx)\cos(\omega t+kx+\theta)\,dt$

$= \frac{1}{\tau}\int_0^\tau \cos(\omega t+kx)\left[\cos(\omega t+kx)\cos\theta-\sin(\omega t+kx)\sin\theta\right]dt$

$= \frac{1}{\tau}\left[\int_0^\tau \cos\theta\cos^2(\omega t+kx)\,dt-\int_0^\tau \cos(\omega t+kx)\sin(\omega t+kx)\right.$

$\left.\sin\theta\,dt\right]$

$= \frac{\cos\theta}{2}-\frac{1}{\tau}\int_0^\tau \cos(\omega t+kx)\sin(\omega t+kx)\sin\theta\,dt$

$\psi = \cos\theta\int_0^\tau \cos(\omega t+kx)\sin(\omega t+kx)\,dt$

$U=\sin(\omega t+kx)$ $\qquad dV=\cos(\omega t+kx)\,dt$

$dU=\omega\cos(\omega t+kx)\,dt$ $\qquad V=\frac{1}{\omega}\sin(\omega t+kx)$

$\psi=\cos\theta\left[\frac{1}{\omega}\sin^2(\omega t+kx)\Big|_0^\tau-\int_0^\tau\cos(\omega t+kx)\sin(\omega t+kx)\,dt\right.$

$=\cos\theta\left[0-\frac{\psi}{\cos\theta}\right]\Rightarrow\psi=-\psi\Rightarrow\psi=0$

$\therefore \frac{1}{\tau}\int_0^\tau \cos(\omega t+kx)\cos(\omega t+kx+\theta)\,dt=\frac{\cos\theta}{2}$ ✓

2-15) a) $X_{MAX} = \frac{F_0}{\omega Z_m}$ $\ni$ $Z_m = \left[R^2 + (\omega m - \frac{K}{\omega})^2\right]^{1/2}$

$$\Rightarrow X_{MAX} = \frac{F_0}{\omega\left[R^2 + (\omega m - K/\omega)^2\right]^{1/2}}$$

X_{MAX} WILL OBTAIN IT'S MAXIMUM VALUE WHEN ωZ_m IS MINIMUM:

$$\begin{aligned}
\omega Z_m &= \omega\sqrt{R^2 + (\omega m - K/\omega)^2} \\
&= \left[\omega^2 (R^2 + [\omega m - \tfrac{K}{\omega}]^2)\right]^{1/2} \\
&= \left[\omega^2 R^2 + \{\omega\ [\omega m - \tfrac{K}{\omega}]\}^2\right]^{1/2} \\
&= \left[\omega^2 R^2 + (\omega^2 m - K)^2\right]^{1/2} \\
&= \left[\omega^2 R^2 + \omega^4 m^2 - 2Km\omega^2 + K^2\right]^{1/2} \\
&= \left[\omega^4 m^2 + (R^2 - 2Km)\omega^2 + K^2\right]^{1/2}
\end{aligned}$$

$$0 = \frac{d(\omega Z_m)}{d\omega} = \tfrac{1}{2}\left[4\omega^3 m^2 + 2(R^2 - 2Km)\omega\right] / \omega Z_m$$

$$= 4\omega^2 m^2 + 2(R^2 - 2Km)$$

$$2\omega^2 m^2 = (2km - R^2)$$

$$\omega = \sqrt{\frac{2km - R^2}{2m^2}} = \sqrt{\frac{K}{m} - \frac{R^2}{2m^2}} = \sqrt{\omega_0^2 - 2\alpha^2} \quad ✓$$

b) $\dot{X}_{MAX} = \frac{F_0}{Z_m}$

$\dot{X}_{MAX}$ WILL OBTAIN IT'S MAXIMUM VALUE WHEN Z_m IS MIN.

$$\begin{aligned}
Z_m &= \left\{R^2 + (\omega m - \tfrac{K}{\omega})^2\right\}^{1/2} \\
&= \left\{R^2 + m^2\omega^2 - 2\omega K + \tfrac{K^2}{\omega^2}\right\}^{1/2}
\end{aligned}$$

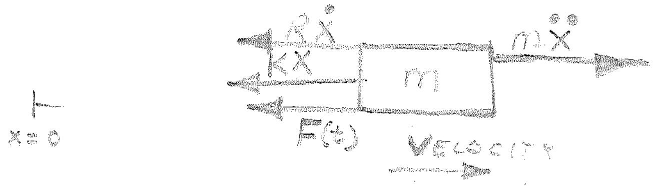
$$0 = \frac{dZ_m}{d\omega} = \left\{2m^2\omega - \frac{2K^2}{\omega^3}\right\} / Z_m$$

$$2m^2\omega = \frac{2K^2}{\omega^3}$$

$$\omega^4 = \frac{K^2}{m^2}$$

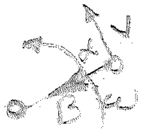
$$\Rightarrow \omega = \omega_0 = \sqrt{K/m} \quad ✓$$

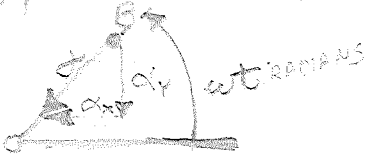
2-16)a)



$$\Sigma F_x = m\ddot{x} = R\dot{x} + kx + F(t)$$
$$= R\dot{x} + kx + F_0 \cos\omega t$$

b)





$\alpha =$ ANGULAR ACCELERATION

$$= \frac{V^2}{B} = \frac{(\omega B)^2}{B} = B\omega^2$$

$$\alpha_x = \alpha \cos\omega t$$
$$= B\omega^2 \cos\omega t$$
$$\Rightarrow F(t) = m a_x = mB\omega^2 \cos\omega t$$
$$\Rightarrow F_0 = \boxed{mB\omega^2} \ ?$$

c) $x = -i \frac{F_0}{\omega} \frac{1}{z} e^{i\omega t}$

AT RESONANCE, $\omega = \omega_r = \sqrt{\frac{k}{m}}$, AND $z = R$

$\Rightarrow x_r = -i \frac{F_0}{\omega_r R} e^{i\omega t}$

$|x_r| = A_1 = \frac{F_0}{\omega_r R}$

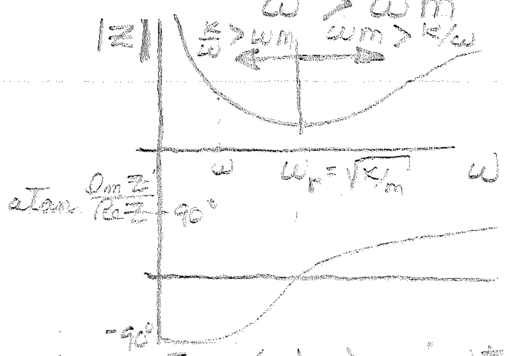
$\Rightarrow \frac{A_1}{B} = \left(\frac{F_0}{\omega_r R}\right)\left(\frac{m\omega_r^2}{F_0}\right) = \frac{m}{R}\omega_r = \frac{m}{R}\sqrt{\frac{k}{m}} = \frac{\sqrt{km}}{R} = Q$

2-17) $\omega m = \frac{3\,kg}{sec}$; $\frac{K}{\omega} = 5\,\frac{kg}{sec}$; $R = \frac{2\,kg}{sec}$

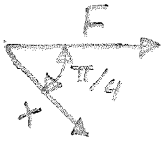
$$Z = R + i\left(\omega m - \frac{K}{\omega}\right)$$
$$= R - i\,2 = 2\sqrt{2}\,e^{-i\pi/4}$$ ✓

DRIVING ω IS LOWER THAN ω_r ✓, BECAUSE

$$\frac{K}{\omega} > \omega m$$



$$x = (-i)\,\frac{F_0}{\omega}\left(\frac{1}{Z_m}\right)e^{i\omega t}$$
$$= \left(e^{-i\frac{\pi}{2}}\right)\frac{F_0}{\omega}\left(\frac{1}{2\sqrt{2}}\,e^{\frac{i\pi}{4}}\right)e^{i\omega t}$$
$$= \frac{F_0}{2\sqrt{2}\,\omega}\,e^{i(\omega t - \pi/4)}$$



THUS, DISPLACEMENT X LAGS FORCE F BY 45° ✓

$$Q = \frac{1}{R}\sqrt{Km}$$
$$KM = \left(\frac{K}{\omega}\right)(\omega m) = 15$$
$$\Rightarrow Q = \frac{1}{2}\sqrt{15} = 1.94$$ ✓

PROBLEMS

3.1 ✓ Any function $y(u)$ where $u = x-ct$ is a solution of the wave equation (3.2) and represents a disturbance being propagated in the +x direction with a speed c. Consider the function

$$y(x,t) = +\sqrt{R^2 - (x-ct)^2}$$

where R=1.5 cm and c = 5 cm/sec, and suppose this function describes the motion of some string. What does the string look like at t=0? At t=1 sec? At t=2 secs? When plotting use same scale for x and y.

3.2. If A, B and α are constants which of the following functions satisfy the wave equation

a) $y(x,t) = Ax + Act + B$

b) $y(x,t) = Ae^{\alpha x - \alpha ct}$

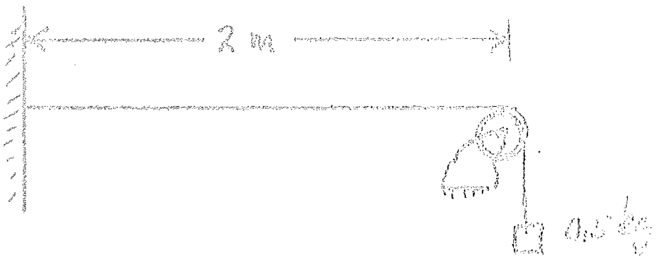
c) $y(x,t) = A(x-ct) + B(x-ct)^2$

d) $y(x,t) = A + B \sin\alpha\,(ct-x)$

3.3. ✓ Show $y(x,t) = A \sin \frac{\omega}{c} x \cos \omega t$ can be written in the form

$$\frac{A}{2}\left\{ \sin\frac{\omega}{c}(x+ct) + \sin\frac{\omega}{c}(x-ct) \right)$$

3.4 ✓ A string is mounted as indicated in the sketch. If pulled aside at some point and released it will be found after a short time to be vibrating approximately in its fundamental mode. If this is the case what would you predict for the frequency of oscillation of a given point on the string if 1 meter length of string has a mass of 0.2 gms? Consider a small piece of the string of 1 mm length. How does the gravitational force on this length compare with the tension forces?



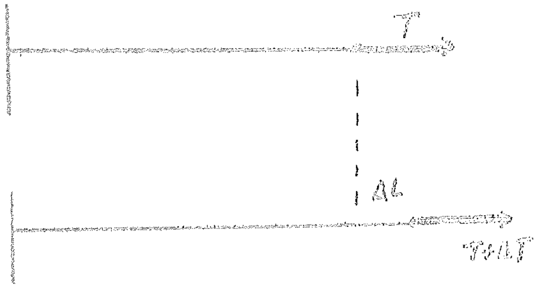
3.5. ✓ Show that

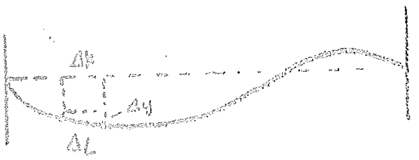
$$\int_0^L \sin\frac{m\pi}{L}x \, \sin\frac{n\pi}{L}x \, dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases} \qquad \begin{matrix} m \, \& \, n \\ \text{integers} \end{matrix}$$

3.6. Find a set of initial conditions that would result in a string vibrating in its fundamental mode.

3.7. ✓ If a string is vibrating in one of its characteristic modes show that the frequency of vibration and the distance D between two successive modes are related by $f = c/2D$ where $c = \sqrt{T/\rho}$ is the velocity of waves on the string.

3.8. If a string has a length L when under a tension T, then increasing the tension a small amount to T + T will cause the string to stretch an amount L. The work done, if second order terms are neglected, is T L. When a string fastened between two rigid supports is vibrating its tension at a given instant of time t_1 is slightly larger than its tension in its rest position. Also its length L' is somewhat larger than its length L in its rest position. One could argue that the potential energy U_p of the string in a given configuration is equal to the work done by the tensile force in stretching the string, i.e.





$$U_p = T\,[L' - L]$$

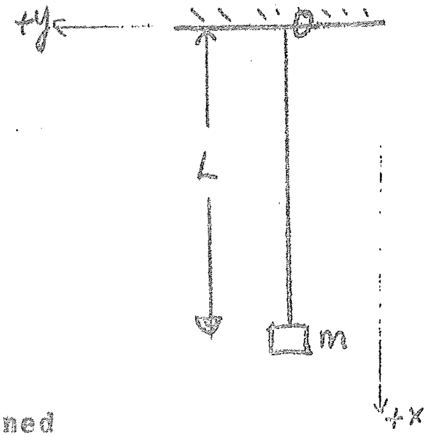
Noting that a typical piece of the string has a length Δx_i in its rest position and some length $\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$

in a given configuration, show that the above expression is equivalent to (3.20).

3.9. The function

$$y(x,t) = \left[C\cos\frac{w}{c}x + A\sin\frac{w}{c}x\right]\cos wt + \left[D\cos\frac{w}{c}x + B\sin\frac{w}{c}x\right]\sin wt$$

is a solution of the wave equation for every positive value of w and for arbitrary values of the constants A, B, C and D. By choosing special values for some of the constants and restricting the values of w this function can be made to satisfy the boundary conditions for a string fastened at both ends. Suppose the string were mounted vertically and a mass M fastened to its lower end as indicated in the figure.

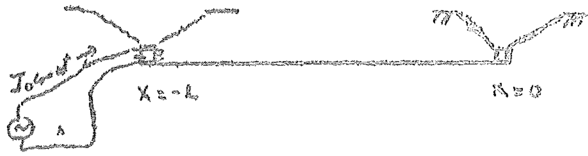


a) If a transverse wave were set up on the string so that the motion of the string were given by some function y(x,t) what condition must this function satisfy at the boundary x=0. What special values must be assigned to the constants A,B,C,D of (1) in order that (1) satisfy this boundary condition?

b) If the string's motion is given by some function y(x,t) then the y-component of the force the string exerts <u>on</u> the mass M can be expressed in terms of T and some derivative of y(x,t). From a knowledge of this force, the y-motion of the mass M can be determined. The y-motion of M and the y-motion of the string at x=L must be identical. This fact leads to a second boundary condition that the function y(x,t) must satisfy at x = L. Write down this condition. Show that (1) can be made to satisfy <u>both</u> boundary conditions by a proper choice of the constants A,B,C,D and by restricting w to special values. Indicate how these eigen values of w could be found.

3.10. For the situation illustrated by the figure shown on the right it was shown that the motion of the string was given by

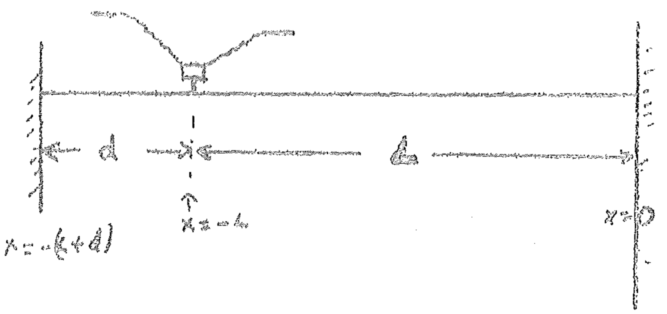
$$y(x,t) = \underset{\sim}{a}\, e^{i(\omega t - kx)} + \underset{\sim}{a}\,\frac{\rho c - \underset{\sim}{Z}_r}{\rho c + \underset{\sim}{Z}_r}$$



where $\underset{\sim}{Z}_r$ is the mechanical impedance of the speaker at x=0. The first term represents a wave being propagated to the right. If it alone were present it would produce a simple harmonic motion of each point of the string. Similarly the second term would if present by itself produce a simple harmonic motion of each point of the string. What would be the phase difference between these separate motions at the point of the string at x=0 when (a) $\underset{\sim}{Z}_r = \infty$, (b) $\underset{\sim}{Z}_r = 0$ (c) $\underset{\sim}{Z}_r = 2\rho c$

3.11. ✓ Show that when the motion of a string is represented by the real part of $\underset{\sim}{y} = \underset{\sim}{a}\, e^{i(\omega t - kx)} + \underset{\sim}{b}\, e^{i(\omega t + kx)}$ the vertical component of the net force on any small element of length dx is $Tk^2 \underset{\sim}{y}\, dx$.

3.12. The arrangement for driving a string shown in Fig. 3.8 is impractical since the voice coil cannot withstand the sidewise force of the string. It is perfectly feasible, however, to drive a string as indicated in the the figure below. Assume the speaker in the figure has a current $I_0 \cos \omega t$ flowing in the voice coil producing a (complex) driving force $BI_0 l e^{i\omega t}$



In the steady state, the waves on the portion of the string to the left of the speaker may be represented by

$$y_l = a_l e^{i(\omega t - kx)} + b_l e^{i(\omega t + kx)}$$

and those on the portion to the right by

$$y_r = a_r e^{i(\omega t - kx)} + b_r e^{i(\omega t + kx)}$$

a) Show by using the boundary conditions that y_l and y_r may be written

$$y_l = A_l \sin k(x+L+d)\, e^{i\omega t}$$

$$y_r = A_r \sin kx\, e^{i\omega t}$$

where

$$A_l = -2 i a_l e^{ik(L+d)} \quad \text{and} \quad A_r = -2 i a_r$$

b) By using the following conditions

$$y_l \Big|_{x=-L} = y_r \Big|_{x=-L}$$

$$\dot{y}_0 = \frac{\partial y_l}{\partial t}\Big|_{x=-L} = \frac{\partial y_r}{\partial t}\Big|_{x=-L}$$

where $\dot{y}_m$ represents the velocity of the voice coil, show that

$$y_l = \frac{BI_0 l \sin k(x+L+d)\, e^{i\omega t}}{i\omega \sin kd\, [Z_m - i\rho c(\cot kd + \cot kL)}$$

$$y_r = \frac{BI_0 l \sin kx\, e^{i\omega t}}{-i\omega \sin kL\, [Z_m - i\rho c(\cot kd + \cot kL)]}$$

where $\underset{\sim}{Z}_m$ is the mechanical impedance of the speaker.

c) If $\underset{\sim}{Z}_m = R + i(\omega m - K/\omega)$ show that the real part of $\underset{\sim}{y}_r$ is

$$y_r = \frac{B I l \sin kx \cos(\omega t - \phi)}{\omega \left\{ \left[\left(\omega m - \frac{K}{\omega}\right) \sin \frac{\omega}{c} L + \rho c \left(\cot \frac{\omega}{c} d \sin \frac{\omega}{c} L + \cos \frac{\omega}{c} L \right) \right]^2 + R^2 \sin^2 \frac{\omega}{c} L \right\}^{\frac{1}{2}}}$$

$\frac{7.7}{8}$

MARKS

✓ Use same scale for both y & x & you get circle

(3-1) $Y(x,t) = [R^2 - (x-ct)^2]^{\frac{1}{2}}$

FOR REAL $Y(x,t)$, $R^2 \geq (x-ct)^2$

IF $R^2 = (x-ct)^2$, $Y(x,t) = 0 \Rightarrow X = ct \pm R$

MAXIMUM DISPLACEMENT @ $\frac{dy}{dx}\Big|_{t=t_0} = 0 \Rightarrow x = ct$

$R = 1.5 \Rightarrow R^2 = 2.25 \text{ cm}^2$; $c = 5 \text{ cm}$

① $t = 0 \Rightarrow Y(x,0) = [R^2 - x^2]^{1/2}$

MAX @ $x = 0$. $Y(R,0) = 0$; $Y(0,0) = R$

② $t = 1 \Rightarrow Y(x,1) = [R^2 - (x-c)^2]^{1/2}$

MAX @ $x = c$; $Y(c+R,1) = 0$; $\boxed{Y(1,0) = \sqrt{R^2 - c^2}}$ compl.

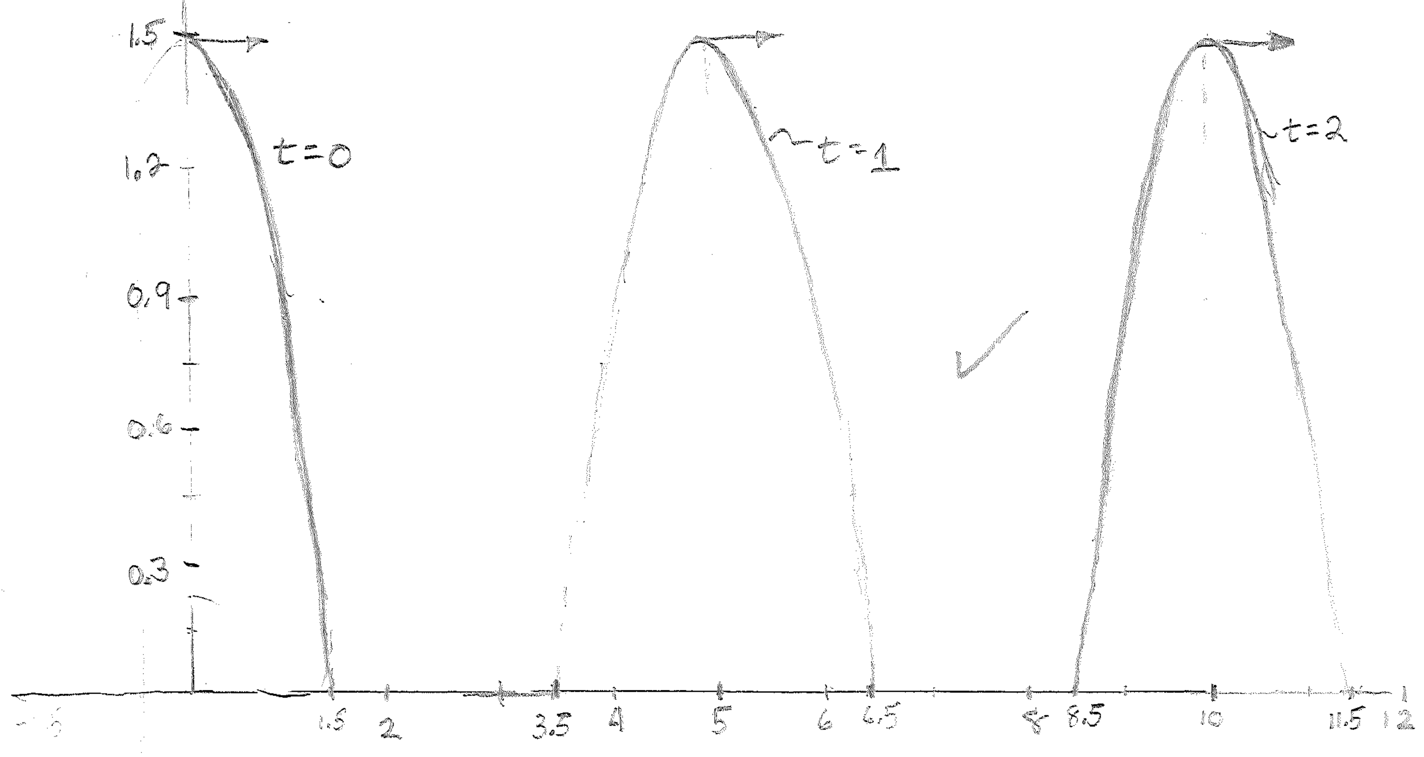
$Y(c-R,1) = 0$

③ $t = 2 \Rightarrow Y(x,2) = [R^2 - (x-2c)^2]^{1/2}$

MAX @ $x = 2c$; $Y(2c+R) = Y(2c-R) = 0$

$Y(2,0) = \sqrt{R^2 - 4c^2}$

THE WAVE IS AN UNDAMPED PARABOLIC PULSE:



✓

3-3) $Y(x,t) = A \cdot \sin\frac{\omega}{c}x \cos\omega t$

$$\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\Rightarrow Y(x,t) = \frac{A}{2}\left[\sin\left\{\frac{\omega}{c}x + \omega t\right\} + \sin\left\{\frac{\omega}{c}x - \omega t\right\}\right]$$

$$= \frac{A}{2}\left[\sin\frac{\omega}{c}(x+ct) + \sin\frac{\omega}{c}(x-ct)\right]$$

✓

3-4) a) A STRING RELEASED WITH ZERO INITIAL VELOCITY WILL VIBRATE IN ACCORDANCE TO:

$$Y(x,t) = A_1 \sin\left(\frac{\pi}{L}x\right) \cos\left(\frac{\pi c}{L}t\right)$$

IN IT'S FUNDEMENTAL MODE.

THE FREQUENCY OF VIBRATION IS:

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\left(\frac{\pi c}{L}\right) = \frac{c}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

FOR THIS SYSTEM: $L = 2\ m$

$$T = mg = (0.5)(9.8) = 4.6\ nt$$

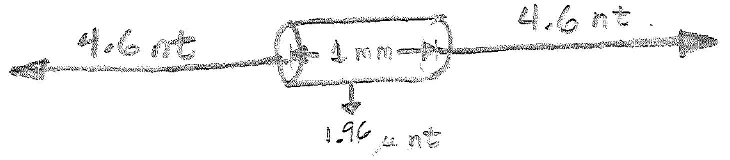
$$\rho = \frac{m}{L} = \frac{2\times10^{-4}}{1} = 2\times10^{-4}\ kg/m$$

$$\Rightarrow f_1 = \frac{1}{4}\sqrt{\frac{4.6}{2\times10^{-4}}} = \frac{1}{4}\sqrt{2.3\times10^{4}} = \frac{1.55}{4}\times10^{2} = 37.8\ Hz$$

b) THE GRAVITATIONAL FORCE ACTING ON 1mm OF THE STRING IS:

$$F_g = \Delta x \rho g = (10^{-3})(2\times10^{-4})(9.8) = 19.6\times10^{-7}\ nt$$

NEGLIGIBLE COMPARED WITH T:



✓

3.5) $\int_0^L \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\,dx$

$$= \int_0^L \frac{1}{2}\left[\cos\left(\frac{m\pi}{L}x - \frac{n\pi}{L}x\right) - \cos\left(\frac{m\pi}{L}x + \frac{n\pi}{L}x\right)\right]$$

$$= \int_0^L \frac{1}{2}\left[\cos\{(m-n)(\tfrac{\pi}{L}x)\} - \cos\{(m+n)(\tfrac{\pi}{L}x)\}\right.$$

FOR $m = n$

$$\int_0^L \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\,dx = \int_0^L \frac{1}{2}\left[1 - \cos\frac{2n\pi}{L}x\right]dx$$

$$= \frac{1}{2}\left[x - \frac{L}{2\pi n}\sin\frac{2n\pi}{L}x\right]_0^L$$

$$= \frac{L}{2}$$ ✓ ($\sin 2n\pi = 0$ FOR $n = 1, 2, ..$)

FOR $m \neq n$

$$\int_0^L \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\,dx = \frac{1}{2}\int_0^L \left[\cos\{(m-n)(\tfrac{\pi}{L}x)\} - \cos\{(m+n)\tfrac{\pi}{L}x\}\right]$$

$$= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin\{(m-n)\tfrac{\pi}{L}x\} - \frac{L(m+n)}{\pi}\sin\{(m+n)\tfrac{\pi}{L}x\}\right]_0^L$$

NOW $\sin 0 = \sin P\pi = 0$ FOR $P = 0, \pm 1, \pm 2, ...$

$\Rightarrow \int_0^L \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\,dx = 0$ FOR $m \neq n$ ✓

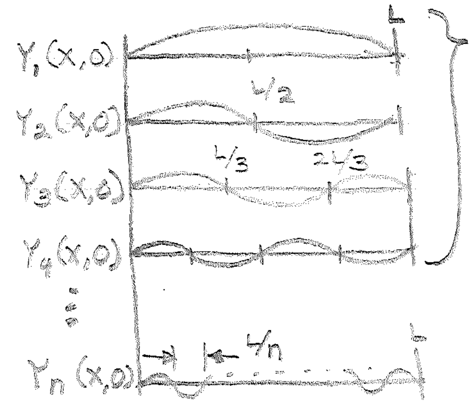
3-7) A STRING RELEASED FROM REST, VIBRATES IN ITS n^{TH} NODE IN ACCORD TO:

$$Y_n(x,t) = A_n \sin \frac{n\pi}{L} x \cos \frac{n\pi c}{L} t$$

THE FREQUENCY OF VIBRATION IS THUS:

$$\omega_n = \frac{n\pi c}{L} \Rightarrow f_n = \frac{nc}{2L} \quad \left(= \frac{\omega_n}{2\pi} \right)$$

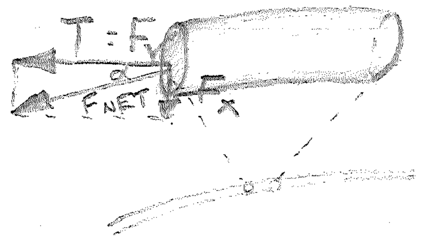
CONSIDER THE NODES OF DIFFERENT MODES



THE DISTANCE BETWEEN NODES IS SEEN TO BE $D = \frac{L}{n}$ ✓

THUS: $f_n = \frac{nc}{2L} = \frac{c}{2}\left(\frac{n}{L}\right) = \frac{c}{2D}$ ✓

3-11) $Y = ae^{i(\omega t - kx)} + be^{i(\omega t + kx)}$



$\alpha << 1$

FOR $\alpha << 1$, $F_{NET} \simeq T$

NOW $F_x = F_{NET} \sin\alpha \simeq T \sin\alpha \simeq T \tan\alpha = T \frac{dY}{dx}$

$\frac{dY}{dx} = -ik[ae^{i(\omega t - kx)} + be^{i(\omega t + kx)}]$

$= -ikY$

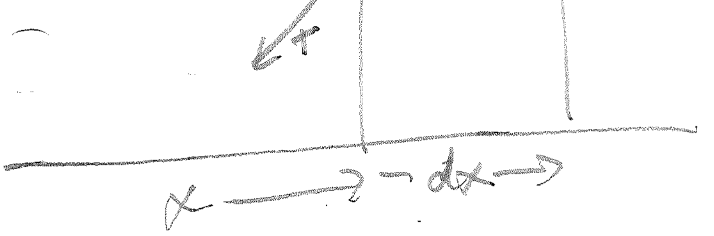
THE FORCE dF_x ACTING ON A STRING SEGMENT OF LENGTH dx WOULD BE:

Not completely clear

$\frac{dF_x}{dx} \cong T \frac{d^2Y}{dX^2}$

$= T(-ik)\frac{dY}{dx}$

$= -Tk^2Y$ $\left(\frac{d^2Y}{dX^2} = -k^2Y\right)$



Net force =

3-4) $Y(x,t) = \left[C \cos \frac{\omega}{c}x + A \sin \frac{\omega}{c}x\right] \cos \omega t$

$\qquad + \left[D \cos \frac{\omega}{c}x + B \sin \frac{\omega}{c}x\right] \sin \omega t$

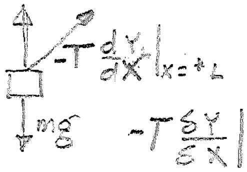
a) $Y(0,t) = 0 = C \cos \omega t + D \sin \omega t$

$\Rightarrow C = D = 0$ ✓

THEN: $Y(x,t) = A \sin \frac{\omega}{c}x \cos \omega t + B \sin \frac{\omega}{c}x \sin \omega t$

$= \left[A \cos \omega t + B \sin \omega t\right] \sin \frac{\omega}{c}x$

b)



$-T \frac{\delta Y}{\delta x}\Big|_{x=+L} = m\ddot{Y}\Big|_{x=+L}$

$-T\frac{\delta Y}{\delta x}\Big|_{x=L} = -T\frac{\omega}{c}\left[A \cos \omega t + B \sin \omega t\right] \cos \frac{\omega L}{c}$

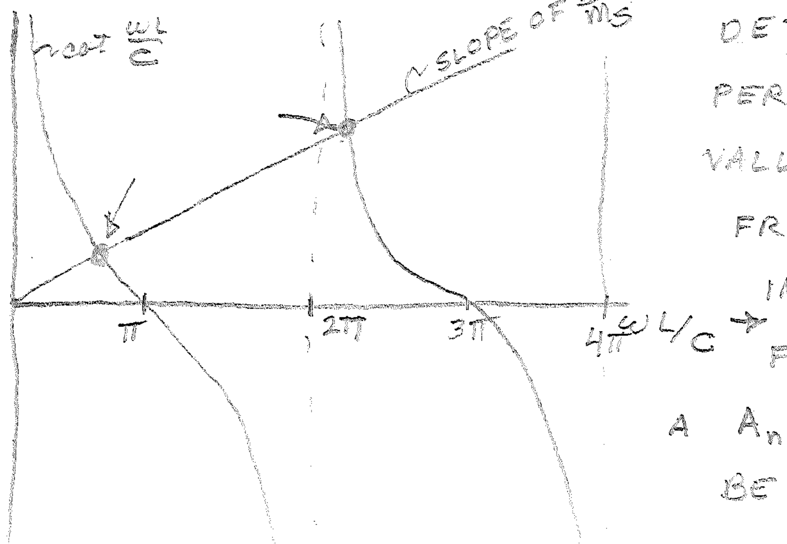
$= m\frac{\delta^2 Y}{\delta t^2}\Big|_{x=L} = -m\omega^2\left[A \cos \omega t + B \sin \omega t\right] \sin \frac{\omega L}{c}$

$\Rightarrow -m\omega \sin \frac{\omega L}{c} = \frac{-T}{c} \cos \frac{\omega L}{c}$

$\cot \frac{\omega L}{c} = \frac{cm\omega}{T}$

$T = \rho c^2 \;;\; \rho = \frac{m_s}{L} \ni m_s =$ MASS OF STRING

$\Rightarrow \cot \frac{\omega L}{c} = \frac{cm\omega}{\rho c^2} = \frac{m\omega}{\rho c} = \frac{m\omega L}{m_s c} = \frac{m}{m_s}\left(\frac{\omega L}{c}\right)$



DETERMINE PERMISSABLE VALUES OF $\frac{\omega L}{c}$ FROM GRAPH INTERSECTIONS, FROM WHICH A A_n AND A B_n MAY BE DETERMINED FROM EACH POINT ✓

3-12) a) $y_\ell = e^{i\omega t}\left[a_\ell e^{-ikx} + b_\ell e^{ikx}\right]$

$y_\ell\left[-(L+d), t\right] = 0 = a_\ell e^{+ik(L+d)} + b_\ell e^{-ik(L+d)}$

$\Rightarrow a_\ell e^{ik(L+d)} = -b_\ell e^{-ik(L+d)}$

$a_\ell = -b_\ell e^{-2ik(L+d)}$

$\Rightarrow Y_\ell = e^{i\omega t}\left[-b_\ell e^{-i2k(L+d)} e^{-ikx} + b_\ell e^{ikx}\right]$

$= b_\ell e^{i\omega t}\left[e^{ikx} - e^{-i2k(L+d)} e^{-ikx}\right]$

$= b_\ell e^{i\omega t} e^{-i k(L+d)}\left[e^{ikx} e^{ik(L+d)} - e^{-i k(L+d)} e^{ikx}\right]$

$= 2i b_\ell e^{-ik(L+d)} e^{i\omega t}\left[\frac{1}{i2}\left\{e^{ik(x+L+d)} - e^{-ik(L+d+x)}\right]\right.$

$= 2i b_\ell e^{-ik(L+d)} e^{i\omega t} \sin k(x+L+d)$

$= A_\ell e^{i\omega t} \sin k(x+L+d)$ ✓

$\Rightarrow A_\ell = 2i b_\ell e^{-ik(L+d)} = 2i\left(-a_\ell e^{+i2k(L+d)}\right) e^{-ik(L+d)}$

$= -2i a_\ell e^{ik(L+d)}$

$Y_r = e^{i\omega t}\left[a_r e^{-kx} + b_r e^{ikx}\right]$

$Y_r(0,t) = 0 = a_r + b_r \Rightarrow b_r = -a_r$

$\Rightarrow Y_r = e^{i\omega t}\left[a_r e^{-kx} - a_r e^{ikx}\right]$

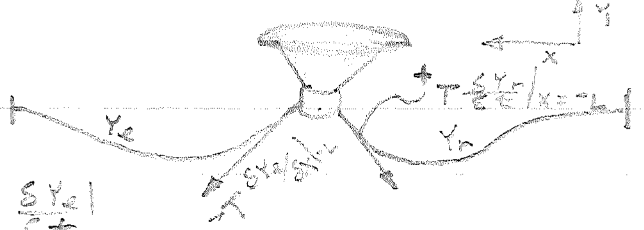
$= -i2a_r e^{i\omega t}\left[\frac{1}{i2}\left\{e^{ikx} - e^{-ikx}\right\}\right]$

$= -i2a_r e^{i\omega t} \sin kx$

$= A_r e^{i\omega t} \sin kx$

$\Rightarrow A_r = -i2a_r$ ✓

b)



$$\dot{Y}_D = \frac{\delta Y_\ell}{\delta t}\Big|_{x=-L} = \frac{\delta Y_r}{\delta t}\Big|_{x=-L}$$

EQUATION OF GOILS MOTION:

$$m\ddot{Y}_0 + R\dot{Y}_D + kY_D = BI_0\ell e^{i\omega t} - T\frac{\delta Y_\ell}{\delta X}\Big|_{x=-L} + T\frac{\delta Y_r}{\delta X}\Big|_{x=-L}$$

$$-\frac{\delta Y_\ell}{\delta X}\Big|_{x=-L} = -A_\ell e^{i\omega t} k \cos kd$$

$$\frac{\delta Y_r}{\delta X}\Big|_{x=-L} = A_r e^{i\omega t} k \cos kL$$

$$m\ddot{Y}_0 + R\dot{Y}_0 + kY_0 = [BI_0\ell - TA_\ell k \cos kd + TA_r k \cos kL]e^{i\omega t}$$

$$\Rightarrow \dot{Y}_D = \frac{1}{Z_m}[BI_0\ell - TA_\ell k \cos kd + TA_r k \cos kL]e^{i\omega t}$$

$$= \frac{\delta Y_\ell}{\delta t}\Big|_{x=-L} = i\omega A_\ell e^{i\omega t} \sin kd$$

$$= \frac{\delta Y_r}{\delta t}\Big|_{x=-L} = i\omega A_r e^{i\omega t} \sin kL$$

$$\Rightarrow A_\ell = A_r \frac{\sin kL}{\sin kd} \;;\; A_r = \frac{\sin kd}{\sin kL} A_\ell$$

$$\Rightarrow [BI_0\ell - TA_\ell k \cos kd + T\frac{\sin kd}{\sin kL} A_\ell k \cos kL]$$

$$= i\omega A_\ell \underline{Z}_m \sin kd$$

$$A_\ell[Tk \cos kd - Tk\frac{\sin kd}{\tan kL} + i\omega Z_m \sin kd]$$

$$= BI_0\ell$$

$$\Rightarrow A_\ell = \frac{BI_0\ell}{Tk[\cos kd + \frac{\sin kd}{\tan kL}] + i\omega Z_m \sin kd}$$

$$\frac{BI_0\ell}{(\sin kd)\,Tk[\cot kd + \cot kL + i\omega Z_m]}$$

$$Y_\ell = A_\ell e^{i\omega t} \sin k(x+L+d)$$

$$= \frac{B I_0 \ell \sin k(x+L+d)\, e^{i\omega t}}{Tk \sin kd \left[i\omega Z_m + (\cot kd + \cot kL)\right]}$$

$$= \frac{B I_0 \ell \sin k(x+L+d)\, e^{i\omega t}}{-i\omega \sin kd \left[Z_m - i\rho c(\cot kd + \cot kL)\right)}$$

$$Y_r = A_r e^{i\omega t} \sin kx$$

$$= A_\ell \frac{\sin kd}{\sin kL} e^{i\omega t} \sin kx$$

$$= \frac{B I_0 \ell \sin kx\, e^{i\omega t}}{-i\omega \sin k\ell \left[Z_m - i\rho c(\cot kd + \cot kL)\right]}$$

c) $$Y_r = \frac{B I_0 \ell \sin kx\, e^{i\omega t}}{-i\omega \sin k\ell \left[R + i(X - \rho c(\cot kd + \cot kL)\right]}$$

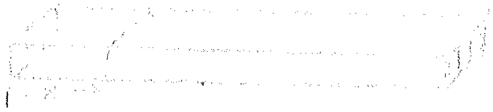
$$= \frac{B I_0 \ell \sin kx \quad e^{i\omega t}}{-iR\omega \sin k\ell + \left[X - \rho c(\cot kd + \cot kL\right]} \qquad \ni X = \omega m - \frac{K}{\omega}$$

$$= \frac{B I_0 \ell \sin kx \,(\cos\omega t + i \sin\omega t)\left[(X - \rho c(\cot kd + \cot kL) + i R\omega \sin k\ell\right.}{\left[X - \rho c(\cot kd + \cot k\ell)\right]^2 + \left[R\omega \sin k\ell\right]^2}$$

$$\Rightarrow \operatorname{Re} Y_r = \frac{B I_0 \ell \sin kx \left[\cos\omega t\,(X - \rho c(\cot kd + \cot kL)) - \sin\omega t\, R\omega \sin k\ell\right.}{\left[X - \rho c(\cot kd + \cot k\ell)\right]^2 + \left[R\omega \sin k\ell\right]^2}$$

$$= \frac{B I \ell \sin kx \cos(\omega t - \phi)}{\omega\left\{\left[(\omega m - \frac{K}{\omega}) \sin \frac{\omega}{c}L + \rho c(\cot \frac{\omega}{c}d \sin \frac{\omega}{c}L + \cos \frac{\omega}{c}L)\right\}^2 + R^2 \sin^2 \frac{\omega}{c}L\right\}^{1/2}}$$ ✓

3. When a rod is subjected only to tensile or compressive forces at its ends, the only stress component different from zero at a point of the rod is $S_{xx} = F/A$.



The three strain components $\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$ are in general all different from zero at a point in the rod being given by

$$\epsilon_{xx} = \frac{1}{Y} S_{xx} \qquad \epsilon_{yy} = -\frac{\sigma}{Y} S_{xx} \qquad \epsilon_{zz} = -\frac{\sigma}{Y} S_{xx}$$

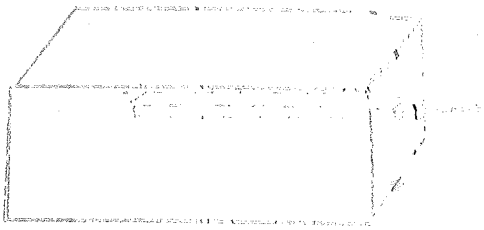
If the applied forces vary with time, then longitudinal waves are set up in the rod and all of the strain components vary with time as well as with x. The derivation of the wave equation indicated that longitudinal waves are propagated with a velocity $c = \sqrt{Y/\rho}$. Now suppose that in addition to the time varying forces applied to the ends of the rods, forces are present at the other faces of the rod and that these forces are at every instant of time of such magnitude and direction to keep the lateral dimensions from changing, i.e. these forces keep ϵ_{yy} and ϵ_{zz} zero at all points of the rod at all times. Show that under these conditions that

$$\text{a)} \qquad S_{yy} = S_{zz} = \frac{\sigma}{1-\sigma} S_{xx}$$

at every point in the rod.

(b) Show that the velocity with which longitudinal waves travel in the rod now depends on σ as well as Y and ρ and find the velocity as a function of σ, Y and ρ. Is this velocity greater or smaller than the velocity $c = \sqrt{Y/\rho}$?

When longitudinal waves are set up in a large block of material by exerting time varying forces on only a small area of its surface as indicated in the figure opposite, then the waves can be thought of as being propagated in a hypothetical rod as indicated in the sketch. The lateral dimensions of this hypothetical rod are prevented from changing by the forces exerted by the material surrounding the rod.



BOB MARKS

$\frac{3}{3}$

1) a) $\epsilon_{YY} = \epsilon_{ZZ} = 0$

$\Rightarrow \quad S_{XX} = (C_1 + C_2)\epsilon_{XX}$

$S_{YY} = C_1 \epsilon_{XX}$

$S_{ZZ} = C_1 \epsilon_{XX}$

$\ni \quad C_1 = \frac{\sigma Y}{(1+\sigma)(1-2\sigma)} \quad ; \quad C_2 = \frac{Y}{1+\sigma}$

SOVLING FOR ϵ_{XX};

$$\epsilon_{XX} = \frac{S_{XX}}{C_1 + C_2} = \frac{S_{YY}}{C_1} = \frac{S_{ZZ}}{C_1}$$

$$\Rightarrow S_{YY} = S_{ZZ} = \frac{C_1}{C_1 + C_2} S_{XX}$$

$$\frac{C_1}{C_1 + C_2} = \frac{\sigma Y}{(1+\sigma)(1-2\sigma)} \left[\frac{1}{\frac{\sigma Y}{(1+\sigma)(1-2\sigma)} + \frac{Y}{1+\sigma}}\right]$$

$$= \frac{\sigma Y}{(1+\sigma)(1-2\sigma)} \left[\frac{(1+\sigma)(1-2\sigma)}{\sigma Y + Y(1-2\sigma)}\right]$$

$$= \frac{\sigma Y}{\sigma Y + Y(1-2\sigma)} = \frac{\sigma Y}{Y(\sigma + 1 - 2\sigma)}$$

$$= \frac{\sigma}{1-\sigma} \quad \checkmark$$

$\therefore \quad S_{YY} = S_{ZZ} = \frac{\sigma}{1-\sigma} S_{XX}$

b) $\epsilon_{XX} = \frac{\delta \xi}{\delta X} = \frac{1}{Y} S_{XX} - \frac{\sigma}{Y} S_{YY} - \frac{\sigma}{Y} S_{ZZ} = \frac{1}{Y} S_{XX} - \frac{2\sigma}{Y} S_{YY}$

$$= \left(\frac{1}{Y} - \frac{2\sigma}{Y}\left[\frac{\sigma}{1-\sigma}\right]\right) S_{XX} = \frac{1}{Y}\left[1 - \frac{2\sigma^2}{1-\sigma}\right] S_{XX}$$

$$\Rightarrow \frac{\delta \xi}{\delta X} = \frac{1}{Y}\left[1 - \frac{2\sigma^2}{1-\sigma}\right]\frac{F_x}{A} = \frac{1}{Y}\left(\frac{1-\sigma-2\sigma^2}{1-\sigma}\right)\frac{F_x}{a} = \frac{(1+\sigma)(1-2\sigma)}{Y(1-\sigma)} \frac{F_x}{A}$$

$$\Rightarrow F_x = \frac{Y(1-\sigma)A}{(1+\sigma)(1-2\sigma)} \frac{\delta \xi}{\delta X} \Rightarrow dF_x = \frac{Y(1-\sigma)A}{(1+\sigma)(1-2\sigma)} \frac{\delta \xi^2}{\delta X^2} dX$$

NEWTON $\Rightarrow \quad (\rho A dx) \frac{\delta^2 \xi}{\delta t^2} = dF_x = \frac{Y(1-\sigma)A}{(1+\sigma)(1-2\sigma)} \frac{\delta^2 \xi}{\delta x^2} dX$

$$\rho \frac{\delta^2 \xi}{\delta t^2} = \frac{Y(1-\sigma)}{(1+\sigma)(1-2\sigma)} \frac{\delta^2 \xi}{\delta x^2}$$

$$\frac{\delta^2 \xi}{\delta t^2} = C_\sigma^2 \frac{\delta^2 \xi}{\delta x^2}$$

$$\ni \quad C_\sigma = \sqrt{\frac{Y}{\rho}} \sqrt{\frac{1-\sigma}{(1+\sigma)(1-2\sigma)}} \quad \checkmark$$

$$0 < \sigma < \frac{1}{2} \Rightarrow \sqrt{\frac{1-\sigma}{(1+\sigma)(1-2\sigma)}} > 1 \quad \checkmark$$

$$\Rightarrow C_\sigma \geq C = \sqrt{\frac{Y}{\rho}}$$

2) $\xi(x,t) = \left[a e^{-ikx} + b e^{ikx}\right] e^{i\omega t}$

1) $F_{x=L} = 0 \Rightarrow \epsilon_{xx}\big|_L = \frac{1}{Y} s_{xx}\big|_L = \frac{F_{xL}}{SY} = 0$

$$\epsilon_{xx}\big|_{x=L} = \frac{\delta \xi}{\delta x}\Big|_{x=L} = 0 = a(-ik)e^{-ikL} + b(ik)e^{ikL}$$

$$\Rightarrow a e^{-ikL} = b e^{ikL} \quad \left(a = b e^{i2kL}\right)$$

2) $F_{x=0} = -F_0 e^{i\omega t} \Rightarrow \epsilon_{xx}\big|_0 = \frac{1}{Y} s_{xx}\big|_0 = \frac{F_0 e^{i\omega t}}{SY}$

$$\Rightarrow \frac{F_0 e^{i\omega t}}{SY} = \epsilon_{xx}\big|_0 = \frac{\delta \xi}{\delta x}\Big|_0 = \left[a(-ik) + b(ik)\right]\cancel{e^{i\omega t}}$$

$$\frac{-iF_0}{SYk} = b - a$$

$$a - b = \frac{iF_0}{SkY} = b\left[e^{i2kL} - 1\right]$$

$$b = \frac{iF_0}{SkY}\,\frac{1}{e^{i2kL} - 1} \qquad a = b e^{i2kL}$$

$$\Rightarrow \xi(x,t) = \frac{iF_0}{SkY}\,\frac{1}{e^{i2kL} - 1}\left[e^{i2kL}e^{-ikx} + e^{ikx}\right]e^{i\omega t}$$

$$= \frac{iF_0}{SkY}\,\frac{1}{e^{ikL} - e^{-ikL}}\left[e^{-ik(x-L)} + e^{ik(x-L)}\right]e^{i\omega t}$$

$$= -\frac{2F_0}{SkY}\,\frac{i}{e^{ikL} - e^{-ikL}}\cos k(x-L)\, e^{i\omega t}$$

$$= \frac{-F_0}{SkY}\,\frac{\cos k(x-L)}{\sin kL}\, e^{i\omega t}$$

LET $K_1 = \frac{-F_0}{SkY \sin kL} = \frac{-F_0}{Skc^2\rho \sin kL}$ ✓

$$\Rightarrow \xi(x,t) = K_1 \cos k(x-L)\, e^{i\omega t}$$

$$Re\{\xi(x,t)\} = K_1 \cos k(x-L) \cos \omega t$$

3) $\epsilon_{xx} = \frac{1}{Y} S_{xx}$

$= \frac{\delta \Xi}{\delta x} = \frac{F_x}{YS}$

$\Rightarrow F_x = YS \frac{\delta \Xi}{\delta X}$ (FROM LEFT TO RIGHT)

$F_x' = -YS \frac{\delta \Xi}{\delta X}$ (FROM RIGHT TO LEFT)

$$Z_s \overset{\triangle}{\equiv} \frac{F_x'}{V_x} = \frac{-YS\, \delta\Xi/\delta x}{\delta\Xi/\delta t} \checkmark$$

FROM PROBLEM 2

$\Xi(x,t) = K_1 \cos k(X-L) e^{i\omega t}$

THEN: $\frac{\delta \Xi}{\delta X} = -K_1 k \sin k(X-L) e^{i\omega t}$

$\frac{\delta \Xi}{\delta t} = K_1 (i\omega) \cos k(X-L) e^{i\omega t}$

$$Z_s = \frac{YS\, \cancel{K_1} k \sin k(X-L)\, \cancel{e^{i\omega t}}}{\cancel{K_1}(i\omega) \cos k(X-L)\, \cancel{e^{i\omega t}}} \checkmark$$

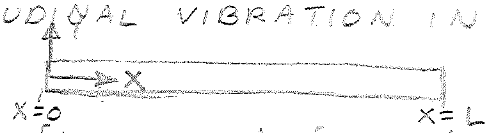
$$= \frac{-iYSk}{\omega} \tan k(X-L) = \frac{iYSk}{\omega} \tan k(L-X)$$

$$= i\rho c S \tan k(L-x)$$

BOB MARKS

$\frac{2}{3}$

LONGITUDINAL VIBRATION IN A FREE ROD



A) $Y(x,t) = [A_1 \cos\alpha x + A_2 \sin\alpha x + A_3 \cosh\alpha x + A_4 \sinh\alpha x] \cos\omega t$
$+ [B_1 \cos\alpha x + B_2 \sin\alpha x + B_3 \cosh\alpha x + A_4 \sinh\alpha x] \sin\omega t$

@ $x = 0$, $\mathcal{M} = 0 \Rightarrow \frac{\delta^2 Y}{\delta x^2}\Big|_{0,t} = 0$

$\frac{\delta^2 Y}{\delta x^2} = -\alpha^2 \Big[\{A_1 \cos\alpha x + A_2 \sin\alpha x - A_3 \cosh\alpha x - A_4 \sinh\alpha x\} \cos\omega t$
$+ \{B_1 \cos\alpha x + B_2 \sin\alpha x - B_3 \cos\alpha x - B_4 \sinh\alpha x\} \sin\omega t\Big]$

$\frac{\delta^2 Y}{\delta x^2}\Big|_{0,t} = 0 = [A_1 - A_3] \cos\omega t + [B_1 - B_3] \sin\omega t$

$\Rightarrow A_1 = A_3 \quad ; \quad B_1 = B_3$

B) $Y(x,t) = [A_1(\cos\alpha x + \cosh\alpha x) + A_2 \sin\alpha x + A_4 \sinh\alpha x] \cos\omega t$
$+ [B_1(\cos\alpha x + \cosh\alpha x) + B_2 \sin\alpha x + B_4 \sinh\alpha x] \sin\omega t$

@ $x = 0$ $F_Y = 0 \Rightarrow \frac{\delta \mathcal{M}}{\delta x}\Big|_{0,t} = 0 \Rightarrow \frac{\delta^3 Y}{\delta x^3} = 0$

$\frac{\delta^3 Y}{\delta x^3} = \alpha^3 \Big[\{A_1(\sin\alpha x + \sinh\alpha x) - A_2 \cos\alpha x + A_4 \cosh\alpha x\} \cos\omega t$
$+ \{B_1(\sin\alpha x + \sinh\alpha x) - B_2 \cos\alpha x + B_4 \cosh\alpha x\} \sin\omega t\Big]$

$\frac{\delta^3 Y}{\delta x^3}\Big|_{0,t} = 0 = [-A_2 + A_4] \cos\omega t + [-B_2 + B_4] \sin\omega t$

$\Rightarrow A_2 = A_4 \quad ; \quad B_2 = B_4$

C) $Y(x,t) = \{A_1[\cos\alpha x + \cosh\alpha x] + A_2[\sin\alpha x + \sinh\alpha x]\} \cos\omega t$
$+ \{B_1[\cos\alpha x + \cosh\alpha x] + B_2[\sin\alpha x + \sinh\alpha x]\} \sin\omega t$

@ $x = L$, $\mathcal{M} = 0 \Rightarrow \frac{\delta^2 Y}{\delta x^2}\Big|_{L,t} = 0$

$\frac{\delta^2 Y}{\delta x^2} = -\alpha^2 \Big[\{A_1(\cos\alpha x - \cosh\alpha x) + A_2(\sin\alpha x - \sinh\alpha x)\} \cos\omega t$
$+ \{B_1(\cos\alpha x - \cosh\alpha x) + B_2(\sin\alpha x - \sinh\alpha x)\} \sin\omega t\Big]$

$\frac{d^2 Y}{d x^2}\Big|_{0,t} = 0 = [A_1(\cos\alpha L - \cosh\alpha L) + A_2(\sin\alpha L - \sinh\alpha L)] \cos\omega t$
$+ [B_1(\cos\alpha L - \cosh\alpha L) + B_2(\sin\alpha L - \sinh\alpha L)] \sin\omega t$

$\Rightarrow A_1(\cos\alpha L - \cosh\alpha L) + A_2(\sin\alpha L - \sinh\alpha L) = 0$

$$\frac{A_2}{A_1} = -\frac{\cos\alpha L - \cosh\alpha L}{\sin\alpha L - \sinh\alpha L} = \frac{B_2}{B_1}$$

D) @ $x = L$, $F_Y = 0 \Rightarrow \frac{\delta M}{\delta x} = 0 \Rightarrow \frac{\delta^3 Y}{\delta x^3} = 0$

$$\frac{\delta^3 Y}{\delta x^3} = \alpha^3 \left[\{A_1(\sin\alpha x + \sinh\alpha x) + A_2(-\cos\alpha x + \cosh\alpha x)\}\cos\omega t \right.$$
$$\left. + \{B_1(\sin\alpha x + \sinh\alpha x) + B_2(-\cos\alpha x + \cosh\alpha x)\}\sin\omega t \right]$$

$$\left.\frac{\delta^3 Y}{\delta x^3}\right|_{L} = 0 = [A_1(\sin\alpha L + \sinh\alpha L) - A_2(\cos\alpha L - \cosh\alpha L)]\cos\omega t$$
$$+ [B_1(\sin\alpha L + \sinh\alpha L) - B_2(\cos\alpha L - \cosh\alpha L)]\sin\omega t$$

$$\Rightarrow A_1(\sin\alpha L + \sinh\alpha L) = A_2(\cos\alpha L - \cosh\alpha L)$$
$$\frac{A_2}{A_1} = \frac{B_2}{B_1} = \frac{\sin\alpha L + \sinh\alpha L}{\cos\alpha L - \cosh\alpha L}$$

$$\frac{A_2}{A_1} = \frac{B_2}{B_1} = \frac{\sin\alpha L + \sinh\alpha L}{\cos\alpha L - \cosh\alpha L} = -\frac{\cos\alpha L - \cosh\alpha L}{\sin\alpha L - \sinh\alpha L}$$
$$-(\sin\alpha L + \sinh\alpha L)(\sin\alpha L - \sinh\alpha L) = (\cos\alpha L - \cosh\alpha L)^2$$
$$-\sin^2\alpha L + \sinh\alpha L = \cos^2\alpha L - 2\cos\alpha L\cosh\alpha L + \cosh^2\alpha L$$
$$2\cos\alpha L\cosh\alpha L = (\cos^2\alpha L + \sin^2\alpha L) + (\sinh^2\alpha L - \cosh^2\alpha L)$$
$$\cos\alpha L\cosh\alpha L = 1$$

$\Rightarrow$ SAME αL VALUES AS IN CLAMPED CASE:

$$\alpha L = \frac{3.01\,\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \sqrt{\frac{\omega}{CI}}\,L$$
$$(\alpha L)_3 \doteq \frac{7\pi}{2} = \sqrt{\frac{\omega}{CI}}\,L \Rightarrow \omega_3 = \left(\frac{7\pi}{2L}\right)^2 CI$$

NOW $Y(x,t) = \left[(\cos\alpha x + \cosh\alpha x) + \frac{A_2}{A_1}(\sin\alpha x + \sinh\alpha x)\right]A_1\cos\omega t$
$+ \left[(\cos\alpha x + \cosh\alpha x) + \frac{B_2}{B_1}(\sin\alpha x + \sinh\alpha x)\right]B_1\sin\omega t$

BUT $\frac{A_2}{A_1} = \frac{B_2}{B_1}$

$$\Rightarrow Y(x,t) = \left[(\cos\alpha x + \cosh\alpha x) + \frac{A_2}{A_1}(\sin\alpha x + \sinh\alpha x)\right] \cdot [A_1\cos\omega t + B_1\sin\omega t]$$

AT THE 3RD EIGEN FREQ $(\alpha L)_3 = \frac{7\pi}{2}$, AND $\omega_3 = \left(\frac{7\pi}{2L}\right)^2 CI$

$$\Rightarrow \left(\frac{A_2}{A_1}\right)_3 = \frac{\sin 7\pi/2 + \sinh 7\pi/2}{\cos 7\pi/2 - \cosh 7\pi/2}$$

$$\sin \frac{7\pi}{2} = 1 \quad ; \quad \cos \frac{7\pi}{2} = 0$$

$$\sinh \frac{7\pi}{2} = \frac{1}{2}\left(e^{7\pi/2} - e^{-7\pi/2}\right)$$

$$= \frac{1}{2}\left(e^{11} - e^{-11}\right)$$

$$\cong \frac{1}{2} e^{11}$$

$$-\cosh \frac{7\pi}{2} = \frac{-1}{2}\left(e^{7\pi/2} + e^{-7\pi/2}\right)$$

$$\simeq \frac{-1}{2} e^{11}$$

$$\Rightarrow \left(\frac{A_2}{A_1}\right)_3 \cong \frac{1 + \frac{1}{2}e^{11}}{-\frac{1}{2}e^{11}} \simeq -1$$

$$\therefore Y_3(x,t) \simeq \left[(\cos \alpha x + \cosh \alpha x) - (\sin \alpha x + \sinh \alpha x)\right]$$

$$\left[A_1 \cos \omega t + B_1 \sin \omega t\right]$$

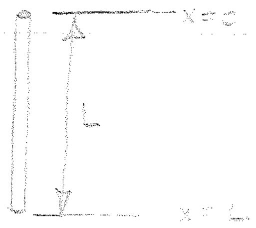
$$\alpha = \frac{7\pi}{2} \frac{1}{L}$$

EIGEN FREQUENCIES IN A FREE ROD

WAVE EQUATION: $\frac{\delta^2 \psi}{\delta t^2} = c^2 \frac{\delta^2 \psi}{\delta x^2} \ni c = \sqrt{\frac{G}{\rho}}$

GENERAL SOLUTION:

$$\psi(x,t) = \left[A_1 \cos \frac{\omega}{c} x + A_2 \sin \frac{\omega}{c} x\right] \cos \omega t$$
$$+ \left[B_1 \cos \frac{\omega}{c} x + B_2 \sin \frac{\omega}{c} x\right] \sin \omega t$$



BOUNDRY CONDITIONS:

$$\tau\Big|_{x=0} = \tau\Big|_{x=L} = 0 \quad ; \tau = G \frac{\pi a^4}{2} \frac{\delta \psi}{\delta x}$$

$$\Rightarrow \frac{\delta \psi}{\delta x}\Big|_{x=0} = \frac{\delta \psi}{\delta x}\Big|_{x=L} = 0$$

$$\frac{\delta \psi}{\delta x} = -\frac{\omega}{c}\left[\left\{A_1 \sin \frac{\omega}{c} x - A_2 \cos \frac{\omega}{c} x\right\} \cos \omega t\right.$$
$$\left. + \left\{B_1 \sin \frac{\omega}{c} x - B_2 \cos \frac{\omega}{c} x\right\} \sin \omega t\right]$$

$$\frac{\delta \psi}{\delta x}\Big|_{x=0} = 0 = A_2 \cos \omega t + B_2 \sin \omega t$$

$$\Rightarrow A_2 = B_2 = 0$$

$$\Rightarrow \psi(x,t) = A_1 \cos \frac{\omega}{c} x \cos \omega t + B_1 \cos \frac{\omega}{c} x \sin \omega t$$
$$= \cos \frac{\omega}{c} x \left[A_1 \cos \omega t + B_1 \sin \omega t\right]$$ ✓

$$\frac{d\psi}{\delta x} = -\frac{\omega}{c} \sin \frac{\omega}{c} x \left[A_1 \cos \omega t + B_1 \sin \omega t\right]$$

$$\frac{\delta \psi}{\delta x}\Big|_{x=L} = 0 \Rightarrow \sin \frac{\omega L}{c} = 0$$

THEN $\frac{\omega L}{c} = n\pi \qquad n = 1, 2, 3, \ldots$ ✓

$$\omega_n = \frac{c}{L}(\pi n)$$
$$= \sqrt{\frac{G}{\rho}} \frac{\pi n}{L}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2L}\sqrt{\frac{G}{\rho}}\, n \qquad n = 1, 2, 3, \ldots$$

Problems for Chapter VI

1. Show that $P(\nu)$ where $\nu = x \sin\theta \cos\phi + y \sin\theta \sin\phi + z \cos\theta$ is a solution of the wave equation 6.5 for arbitrary values of θ and ϕ.

2. The density ρ at a point in any medium is defined as the ratio of the mass contained in a tiny volume surrounding the point to this tiny volume. A particle of a fluid is thought of as a fixed mass of fluid which occupies some tiny volume V when the pressure is P. If the pressure increases to some value P', the volume occupied by the fixed mass will shrink to a value V' and consequently the density of the fluid at the point where the particle is located will change to a value ρ'. When an harmonic wave exists in a fluid the density ρ' varies slightly above and below some mean value ρ and the quantity $S = (\rho' - \rho)/\rho$ is called the condensation at the point. The density ρ' at any instant is only slightly different from ρ and $S \ll 1$. Show that the acoustic pressure at a point and the condensation at a point are related by $\mathcal{P} = B_a S$ where B_a is the adiabatic bulk modulus.

3. The stress-strain relation (6.2) can be written in vector notation as

$$\mathcal{P} = -B_a \operatorname{div} \vec{s} \qquad \text{(i)}$$

where $\vec{s}$ is the particle displacement vector with components ξ, η, ζ. Similarly the three equations of (6.3) and (6.4) can be written as the single vector equation

$$-\overrightarrow{\operatorname{grad}}\, \mathcal{P} = \rho \frac{\partial^2 \vec{s}}{\partial t^2} \qquad \text{(ii)}$$

3. (continued)

By taking the divergence of both sides of this equation and substituting from (i) one obtains the wave equation

$$c^2 \nabla^2 P = \frac{\partial^2 P}{\partial t^2} \qquad c = \sqrt{B/\rho}$$

where $\nabla^2 P = \overrightarrow{\text{grad}}\ \text{div}\ P$. In cylindrical coordinates the gradient of any scalar point function such as P is

$$\overrightarrow{\text{grad}}\ P = \frac{\partial P}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial P}{\partial \phi}\hat{\phi} + \frac{\partial P}{\partial z}\hat{k}$$

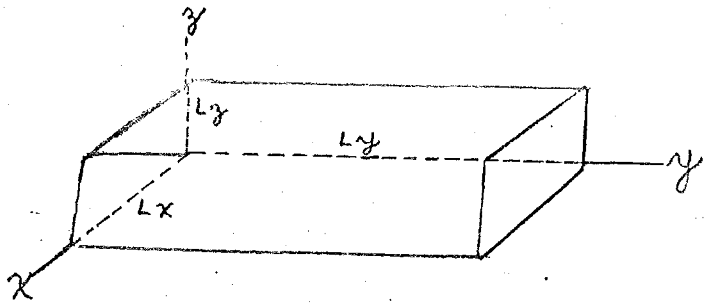
where $\hat{r}$, $\hat{\phi}$ and $\hat{k}$ are unit vectors in the r, Ø and z-direction respectively. Also for any vector $\vec{E}$ whose r, Ø and z components are E_n, E_ϕ, and E_z respectively

$$\text{div}\ \vec{E} = \frac{\partial E_r}{\partial r} + \frac{E_r}{r} + \frac{1}{r}\frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$$

Using these expressions write down the wave equation in cylindrical coordinates.

4. Given $\alpha = 3m^{-1}$, $\beta = 4m^{-1}$, $\gamma = 5m^{-1}$. Find the directions of propagation of the waves represented by each of the eight terms of (6.12).

5. Suppose a gas confined in a rigid box of dimensions L_x, L_y, L_z is vibrating in a characteristic mode for which $n_x = 1$, $n_y = 1$, $n_z = 1$. At any point of the box the acoustic pressure varies harmonically with an amplitude A, which in general is different at different points. If one measured this amplitude at various points with a microphone, at which points would one find the largest amplitude?



6. Find the posxtions of the nodal planes for a fluid confined in a rigid box of dimensions L_x, L_y, L_z and vibrating in a characteristic mode for which $n_x = 2$, $n_y = 1$, $n_3 = 1$.

7. The wave equation in cylindrical coordinates is

$$c_0^2\left[\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial z^2}\right] = \frac{\partial^2 p}{\partial t^2}$$

(i) Show by using the separation of variables approach that one can obtain an harmonic solution of the form

$$P(r, \phi, z) = J_m(\sqrt{k^2-\alpha^2}\, r)\left[A_1\cos m\phi + B_1 \sin m\phi\right]\left[A_2 \cos\alpha z + B_2 \sin\alpha z\right]\cdot\left[A_3 \cos\omega t + B_3\, \mathrm{sim}\,\omega t\right]$$

where m is any positive integer including zero, and A_1, A_2, A_3, B_1, B_2, B_3 α and ω are arbitrary (subject to the restriction that $k^2-\alpha^2 > 0$). Here $k = \frac{\omega}{c}$, and $J_m(\sqrt{k^2-\alpha^2}\, r)$ is Bessel's function of order m.

(ii) Consider a cylindrical cavity of length L and radius a. If the walls of this cavity are rigid so that the component of the particle displacement perpendicular

7. (continued)

to the walls must be zero, show that the harmonic solution will satisfy the boundary conditions at $z = 0$ and $z = L$ only if

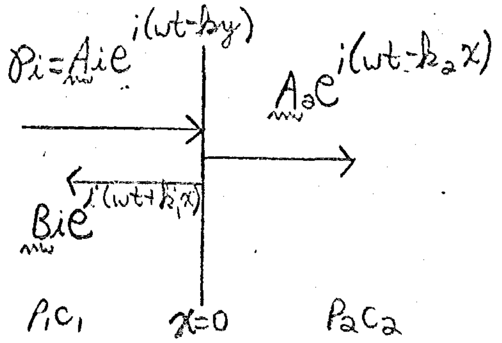
$$B_2 = 0 \quad \text{and} \quad \alpha = \frac{n\pi}{L} \qquad n = 0, 1, 2, 3, \ldots$$

For any pair of allowed values of m and n there will be an harmonic solution satisfying <u>all</u> boundary conditions only for certain special values of ω (and $f = \omega/2\pi$). Find some of these eigen frequencies for the following two cases (1) m=0, n=0; (2) m=1, n=1.

8. For any wave guide, the cut-off frequency for any mode is the lowest frequency f for which the mode can exist in the guide. For air at 20°C and a rectangular wave guide of dimensions $L_x = 0.05$m, $L_y = 0.10$m what is the cut-off frequency for the mode characterized by $n_x = 1$, $n_y = 1$?

9. Show that for normal incidence the requirement that the specific acoustic impedance $\underset{\sim}{z} = \underset{\sim}{P}/\underset{\sim}{u}$ be continuous across a boundary separating the fluids leads to

$$\frac{\underset{\sim}{B_1}}{\underset{\sim}{A_1}} = \frac{\rho_2 c_2/\rho_1 c_1 - 1}{\rho_2 c_2/\rho_1 c_1 + 1}$$



for the ratio of the amplitude of the reflected wave to that of the incident wave. Determine A_2/A_1 from the above ratio and the requirement that the average rate energy is brought to the surface by the incident wave is equal to that carried away by the reflected and refracted waves. Calculate the sound power transmission coefficient.

10. When as in the figure for problem 9, a plane wave is incident on a boundary separating two fluids, the reflected wave is said to suffer a phase shift of 180° if the harmonic variations of the pressure produced by the incident and reflected waves separately are 180° out of phase at the boundary. Under what conditions will such a phase shift occur? When it does occur, is there also a phase difference of 180° between the particle velocity at the boundary due to the incident wave alone, and the particle velocity at the boundary due to the reflected wave alone?

BOB MARKS

6.4/8

2) $$\boxed{\frac{\Delta \rho}{\Delta V} = \frac{\rho_1 - \rho}{V_1 - V} = \overset{?}{m} = \frac{\rho}{V}}\ ?$$

$$\Rightarrow \frac{\Delta V}{V} = \frac{\Delta \rho}{\rho} = \frac{\rho_1 - \rho}{\rho} = S$$

Now $$B_a \overset{\Delta}{=} -\left(\frac{\Delta P}{\Delta V}\right) V$$

$$= -\Delta P \left(\frac{V}{\Delta V}\right)$$

$$= -(P_1 - P)\left(\frac{V}{\Delta V}\right)$$

$$= P/S$$

$$\Rightarrow P = B_a S$$

$$\frac{\rho_1 - \rho}{V_1 - V} = -\frac{\rho_1}{V} \approx -\frac{\rho}{V} \quad \text{since } \frac{\rho_1}{\rho} << 1$$

However this is not equal to m

$$\frac{\Delta \rho}{\rho} = -\frac{\Delta V}{V}$$

.8 ②

1.

1

1

.9

.9

1.0

3) $c^2 \nabla^2 \rho = \frac{\delta^2 \rho}{\delta t^2}$

$$grad\ \rho = \nabla \times \rho = \frac{\delta \rho}{\delta r}\hat{r} + \frac{1}{r}\frac{\delta \rho}{\delta \phi}\hat{\phi} + \frac{\delta \rho}{\delta z}\hat{k}$$

$$div\ (grad\ \rho) = \left[\left(\frac{\delta}{\delta r} + \frac{1}{r}\right)\hat{r}, \left(\frac{1}{r}\frac{\delta}{\delta \phi}\right)\hat{\phi}, \frac{\delta}{\delta z}\hat{z}\right] \cdot \left[\frac{\delta \rho}{\delta r}\hat{r}, \frac{1}{r}\frac{\delta \rho}{\delta \phi}\hat{\phi}, \frac{\delta \rho}{\delta z}\boxed{\hat{k}}\right]$$

$$= \left(\frac{\delta}{\delta r} + \frac{1}{r}\right)\frac{\delta \rho}{\delta r} + \frac{1}{r}\frac{\delta}{\delta \phi}\left(\frac{1}{r}\frac{\delta \rho}{\delta \phi}\right) + \frac{\delta}{\delta z}\left(\frac{\delta \rho}{\delta z}\right)$$

$$= \frac{\delta^2 \rho}{\delta r^2} + \frac{1}{r}\frac{\delta \rho}{\delta r} + \frac{1}{r^2}\frac{\delta^2 \rho}{\delta \phi^2} + \frac{\delta^2 \rho}{\delta z^2}$$

$$\Rightarrow c^2\left[\frac{\delta^2 \rho}{\delta r^2} + \frac{1}{r}\frac{\delta \rho}{\delta r} + \frac{1}{r^2}\frac{\delta^2 \rho}{\delta \phi^2} + \frac{\delta^2 \rho}{\delta z^2}\right] = \frac{\delta^2 \rho}{\delta t^2}$$

✓

5) $\mathcal{P}_{n_x n_y n_z}(x, y, z) = \cos\frac{n_x\pi}{L_x} x \cos\frac{n_y\pi}{L_y} y \cos\frac{n_z\pi}{L_z} z$

$\cdot\left[A_{n_x n_y n_z} \cos(\omega_{n_x n_y n_z} t + \Omega_{n_x n_y n_z})\right]$

$\Rightarrow \mathcal{P}_{111}(x, y, z) = \cos\frac{\pi}{L_x} x \cos\frac{\pi}{L_y} y \cos\frac{\pi}{L_z} z \left[A_{111} \cos(\omega_{111} t + \Omega_{111})\right]$

THE AMPLITUDE $A_{111}(x, y, z)$ OF THE HARMONIC VARIATIONS OF PRESSURE @ x, y, z IN THE BOX:

$\mathcal{P}_{111}(x, y, z) = A_{111}(x, y, z) \cos(\omega_{111} t + \Omega_{111})$

$\Rightarrow A_{111}(x, y, z) = A_{111} \cos\frac{\pi}{L_x} x \cos\frac{\pi}{L_y} y \cos\frac{\pi}{L_z} z$

$[A_{111}(x, y, z)]_{MAX} = A_{111}$

RECOGNIZE: $0 \leq x \leq L_x$, $0 \leq y \leq L_y$; $0 \leq z \leq L_z$

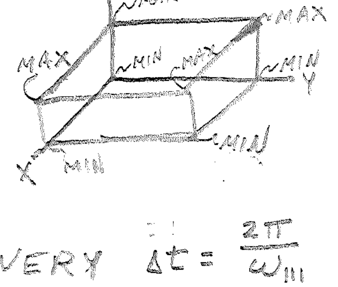
$\mathcal{P}$ IS MAXIMUM WHEN $\overset{THE}{\vee}$ COSINE TERM PRODUCTS ARE UNITY, OR AT:

$@\left(t = \left[\frac{2\pi - \Omega_{111}}{\omega_{111}}\right] n\right)$

X	Y	Z	$\times A_{11}$
0	0	0	$(1 \times 1 \times 1) = 1$
L_x	L_y	0	$(-1 \times -1 \times 1) = 1$
L_x	0	L_z	$(-1 \times 1 \times -1) = 1$
0	L_x	L_z	$(1 \times -1 \times -1) = 1$

MINIMUMS WILL OCCUR WHEN THE COSINE TERM'S PRODUDUCTS ARE MINUS UNITY: $\left(@ t = \left[\frac{2\pi - \Omega_{111}}{\omega_{111}}\right] n\right)$

X	Y	Z	$\times A_{11}$
0	L_y	0	$(1 \times -1 \times 1) = -1$
L_x	0	0	$(-1 \times 1 \times 1) = -1$
0	0	L_y	$(1 \times 1 \times -1) = -1$
L_x	L_y	L_z	$(-1 \times -1 \times -1) = -1$



THE MAX & MIN VALUES WILL SWITCH SIGNS EVERY $\Delta t = \frac{2\pi}{\omega_{111}}$

$\Rightarrow$ MAXIMUM $\mathcal{P}$ AMPLIDUDE IS AT THE BOX'S CORNERS.

$= A_{111}$

6) $P_{n_x n_y n_z}(x, y, z) = A_{n_x n_y n_z} \cos \frac{n_x \pi}{L_x} X \cos \frac{n_y \pi}{L_y} Y \cos \frac{n_z \pi}{L_z} Z \cos(\omega_{n_x n_y n_z} t + \vartheta_{n_x n_y n_z})$

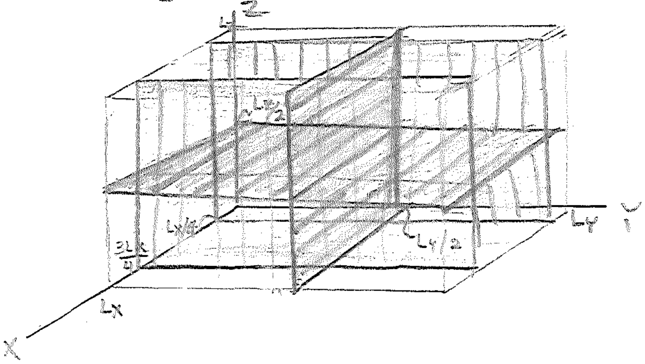
$\Rightarrow P_{211}(x, y, z) = A_{211} \cos \frac{2\pi}{L_x} X \cos \frac{\pi}{L_y} Y \cos \frac{\pi}{L_z} Z \cos(\omega_{211} t + \Omega_{211})$

$0 \le X \le L_x \; ; \; 0 \le Y \le L_y \; ; \; 0 \le Z \le L_z$

$\cos \frac{2\pi}{L_x} X = 0 @ X = \frac{L_x}{4}, \frac{3L_x}{4}$

$\cos \frac{\pi}{L_y} Y = 0 @ Y = \frac{L_y}{2}$

$\cos \frac{\pi}{L_z} Z = 0 @ Z = \frac{L_z}{2}$



ACOUSTIC PRESSURE
NODAL PLANES

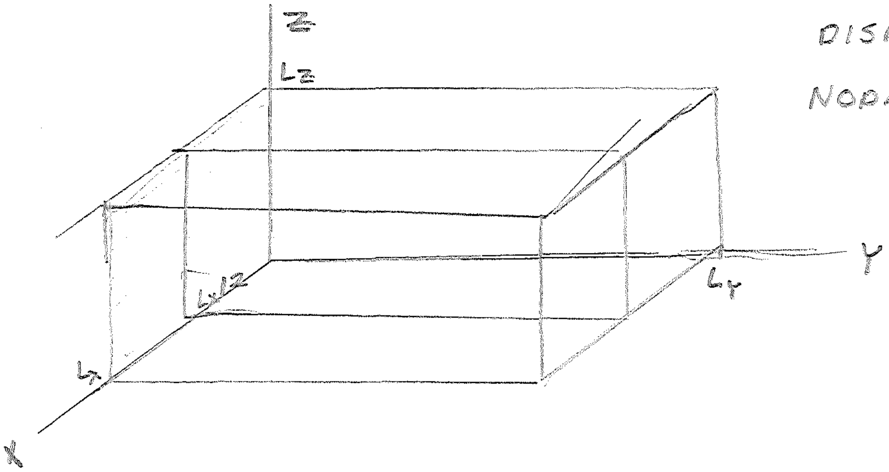
DISPLACEMENTS (ξ, η, ζ) ARE PROPORTIONAL TO $\left(\frac{\delta P}{\delta X}, \frac{\delta P}{\delta Y}, \frac{\delta P}{\delta Z}\right)$

$\Rightarrow$ DISPLACEMENT NODES WILL APPEAR WHEN;

$\sin \frac{2\pi}{L_x} X = 0 \Rightarrow X = 0, \frac{L_x}{2}, L_x$

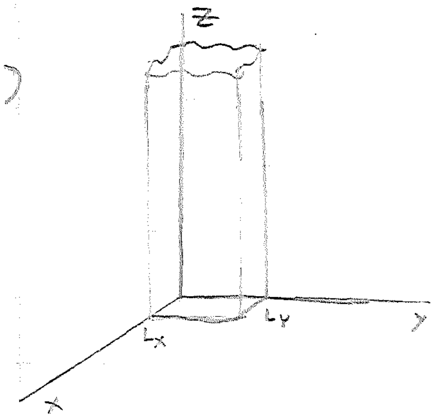
$\sin \frac{\pi}{L_y} Y = 0 \Rightarrow Y = 0, L_y$

$\sin \frac{\pi}{L_z} Z = 0 \Rightarrow Z = 0, L_z$



DISPLACEMENT
NODAL PLANES

5)



$$P(x,y,z,t) = A\cos(\alpha x+\Omega_x)\cos(\beta y+\Omega_y)\cos(\gamma z+\Omega_z)$$
$$\cdot\cos(\omega t+\Omega_t)$$

BOUNDRY CONDITIONS: $P_{0,y,z,t} = P_{L_x,y,z,t} = P_{x,0,z,t} = P_{x,L_y,z,t} = 0$

YIELD: $P(x,y,z,t) = A'\cos\frac{n_x\pi}{L_x}x\cos\frac{n_y\pi}{L_y}y\cos(\gamma z+\Omega_z)\cos(\omega t+\Omega_t)$

$$= \frac{A'}{2}\cos\frac{n_x\pi}{L_x}x\cos\frac{n_y\pi}{L_y}y\Big[\cos(\gamma z+\omega t+\Omega_z+\Omega_t)$$
$$+\cos(\gamma z-\omega t+\Omega_z-\Omega_t)\Big]$$

$$= \frac{A'}{2}\cos\frac{n_x\pi}{L_x}x\cos\frac{n_y\pi}{L_y}y\Big[\cos\gamma\Big(z+\frac{\omega}{\gamma}t+\frac{\Omega_z+\Omega_t}{\gamma}\Big)$$
$$+\cos\gamma\Big(z-\frac{\omega}{\gamma}t+\frac{\Omega_z-\Omega_t}{\gamma}\Big)\Big]$$

$c' = \omega/\gamma$

NOW $\alpha^2+\beta^2+\gamma^2 = \left(\frac{\omega}{c}\right)^2$

$$\Rightarrow \quad \gamma = \left[\left(\frac{\omega}{c}\right)^2-\beta^2-\alpha^2\right]^{1/2}$$
$$= \left[\left(\frac{\omega}{c}\right)^2-\left(\frac{n_x\pi}{L_x}\right)^2-\left(\frac{n_y\pi}{L_y}\right)^2\right]^{1/2}$$

$\Rightarrow$ FOR γ TO BE REAL:

$$\left(\frac{\omega}{c}\right)^2 > \left(\frac{n_x\pi}{L_x}\right)^2+\left(\frac{n_y\pi}{L_y}\right)^2$$

CUT OFF FREQUENCY IS DEFINED AS THE LIMITING CASE: $\omega_c = c\sqrt{\left(\frac{n_x\pi}{L_x}\right)^2+\left(\frac{n_y\pi}{L_y}\right)^2}$

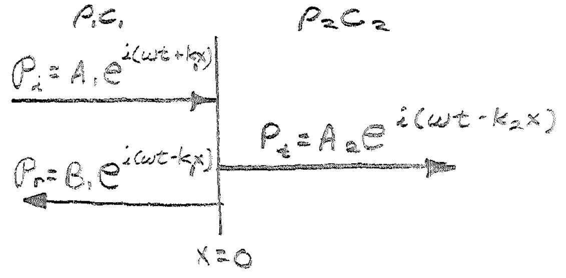
FOR $n_x = n_y = 1$; $L_x = 0.05$; $L_y = 0.1$

$$2\pi f_c = \omega_c = c\pi\boxed{\sqrt{25+10}} = c\pi\sqrt{\left(\frac{1}{.05}\right)^2+\left(\frac{1}{.1}\right)^2} = c\pi\sqrt{500}$$

$\Rightarrow f_c = \frac{c}{2}\cancel{\sqrt{35}}$ /SEC, WHERE c = VEL. OF SOUND @ 20°C $\cong \overset{343}{300}\ \frac{m}{SEC}$

$\therefore f_c =$ ~~$150\sqrt{35}$~~ /SEC $\cong$ 3,840 hertz

9)



$$@_{x=0} \quad P_i = A_i e^{i\omega t} \; ; \; P_r = B_1 e^{i\omega t}$$

$$\Rightarrow P_L = A_1 e^{i\omega t} + B_1 e^{i\omega t}$$

$$\mu_L = \frac{\delta \xi_L}{\delta t} = \frac{A_1}{\rho_1 c_1} e^{i\omega t} - \frac{B_1}{\rho_1 c_1} e^{i\omega t}$$

$$Z_L = \frac{P_L}{\mu_L} = \rho_1 c_1 \left[\frac{A_1 + B_1}{A_1 - B_1}\right]$$

$$P_R\Big|_0 = A_2 e^{i\omega t}$$

$$\mu_R\Big|_{x=0} = \frac{A_2}{\rho_2 c_2} e^{i\omega t}$$

$$\Rightarrow Z_R = \frac{P_r}{U_r} = \rho_2 c_2 \quad \checkmark$$

$$Z_L = Z_r$$

$$\Rightarrow \rho_1 c_1 \left[\frac{A_1 + B_1}{A_1 - B_1}\right] = \rho_2 c_2$$

$$\frac{A_1 + B_1}{A_1 - B_1} = \frac{\rho_2 c_2}{\rho_1 c_1}$$

$$\therefore \frac{B_1}{A_1} = \frac{\rho_2 c_2 / \rho_1 c_1 - 1}{\rho_2 c_2 / \rho_1 c_1 + 1} \quad \checkmark$$

ENEGRY CONSIDERATIONS:

$$\frac{A_1^2}{2\rho_1 c_1} ds \cos\phi_1 - \frac{|B_1|^2}{2\rho_1 c_1} ds \cos\phi_1 = \frac{|A_2|^2}{2\rho_2 c_2} ds \cos\phi_2$$

$$\phi_1 = \phi_2 = 0$$

$$\Rightarrow \frac{1}{2\rho_1 c_1}\left[A_1^2 - B_1^2\right] = \frac{1}{2\rho_2 c_2} A_2^2$$

$$\Rightarrow \frac{|A_2|^2}{|A_1|^2} = \frac{\rho_2 c_2}{\rho_1 c_1}\left[1 - \frac{|B_1|^2}{|A_1|^2}\right]$$

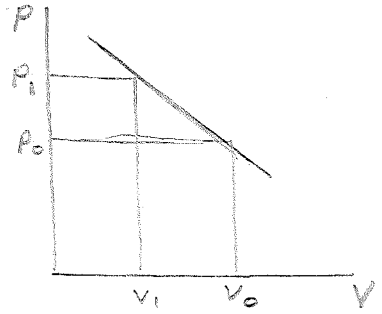
$$= \frac{\rho_2 c_2}{\rho_1 c_1}\left[1 - \frac{(\rho_2 c_2/\rho_1 c_1 - 1)^2}{(\rho_2 c_2/\rho_1 c_1 + 1)^2}\right]$$

$$\therefore \alpha_t \triangleq \frac{\rho_1 c_1}{\rho_2 c_2}\frac{A_1^2}{A_2^2} = \left(\frac{\rho_1 c_1}{\rho_2 c_2}\right)^2 \left[\cancel{1 - \frac{(\rho_2 c_2/\rho_1 c_1 - 1)^2}{(\rho_2 c_2/\rho_1 c_1 + 1)^2}}\right]^{-1}$$

$$= r_{21}^2 \Big/ \left(1 - \frac{(r_{21} - 1)^2}{(r_{21} + 1)^2}\right) = \left[r_{12}^2\left(1 - \frac{(r_{21} - 1)^2}{(r_{21} + 1)^2}\right)\right]^{-1}$$

$$= \left[r_{12}^2 - \left(\frac{1 - r_{12}}{1 + r_{12}}\right)^2\right]^{-1}$$

11)



$$P - P_0 = \frac{P_0 - P_1}{V_0 - V_1}(V - V_0)$$

$$\Rightarrow P = \frac{P_0 - P_1}{V_0 - V_1}(V - V_0) + P_0$$

$$dE = \int_{V_1}^{V_0} P\,dV$$
$$= \int_{V_1}^{V_0}\left[\left(\frac{P_0 - P_1}{V_0 - V_1}\right)(V - V_0) + P_0\right]dV$$
$$= \frac{P_0 - P_1}{V_0 - V_1}\left(\frac{V^2}{2} - V_0 V\right)\Big|_{V_1}^{V_0} + P_0 V\Big|_{V_1}^{V_0}$$
$$= \frac{P_0 - P_1}{V_0 - V_1}\left[\frac{V_0^2 - V_1^2}{2} - V_0(V_0 - V_1)\right] + P_0(V_0 - V_1)$$
$$= (P_0 - P_1)\left[\frac{V_0 + V_1}{2} - V_0\right] + P_0(V_0 - V_1)$$
$$= (P_0 - P_1)\left(\frac{-V_0 + V_1}{2}\right) + P_0(V_0 - V_1)$$
$$= (P_1 - P_0)\left(\frac{V_0 - V_1}{2}\right) + P_0(V_0 - V_1)$$
$$= \frac{\mathcal{P}}{2}(V_0 - V_1) + P_0(V_0 - V_1)$$
$$= \left(P_0 + \frac{\mathcal{P}}{2}\right)(V_0 - V_1)$$

b) $\mathcal{P}^2 = A^2 \cos^2(\omega t - kx)$

$$\Rightarrow \frac{\mathcal{P}^2}{2B_a} = \frac{A^2 \cos^2(\omega t - kx)}{2B_a}$$

$$\frac{dE_{AV}}{V_0} = \left(\frac{\mathcal{P}^2}{2B_a}\right)_{AVE} = \frac{A^2}{2B_a}\int_T \cos^2(\omega t - kx)\,dt \qquad \ni T = \frac{2\pi}{\omega}$$
$$= \frac{A^2}{4B_a} = \frac{A^2}{4\rho c^2} \qquad (B_a = \rho c^2)$$

c) $-\frac{\delta \mathcal{P}}{\delta x} = \rho \frac{\delta^2 \xi}{\delta t^2} \Rightarrow \frac{\delta^2 \xi}{\delta t^2} = \frac{-1}{\rho} A k \sin(\omega t - kx) = \frac{-\omega A}{\rho c}\sin(\quad)$

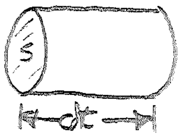
$$\Rightarrow \frac{\delta \xi}{\delta t} = \frac{A}{\rho c}\cos(\omega t - kx)$$

$$U^2_{AVE} = \left[\left(\frac{\delta \xi}{\delta t}\right)^2\right]_{AVE} = \frac{A^2}{2\rho^2 c^2} \Rightarrow \frac{1}{2}\rho U_{AV}^2 = \frac{A^2}{4\rho c^2}$$

$$\Rightarrow \left(\frac{\mathcal{P}^2}{2B_a}\right)_{AV} + \frac{1}{2}\rho U_{AV}^2$$
$$= \frac{A^2}{4\rho c^2} + \frac{A^2}{4\rho c^2} = \frac{A^2}{2\rho c^2}$$

d) $I = \left[\frac{(dE)_{AVE}}{dt}\right]\frac{1}{S}$ = AVERAGE ENERGY FLOW THRU UNIT ARE NORMAL TO PROPAGATON DIRECTION

$$\frac{\boxed{dE}_{AVE}}{V_o} = \frac{A^2}{2\rho c^2}$$

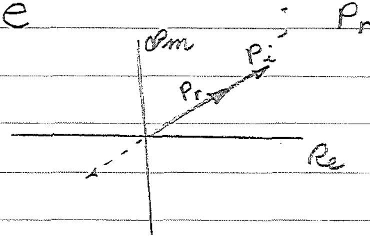


$V_o = Scdt$

$$\Rightarrow [dE]_{AVE} = \frac{A^2 S c dt}{2\rho c^2} = \frac{A^2 S dt}{2\rho c}$$

$$\Rightarrow I = \frac{1}{S}\frac{[dE]_{AVE}}{dt} = \frac{A^2}{2\rho c}$$ ✓

10) $P_i = A_1 e^{i(\omega t + kx)}$; $P_r = B_1 e^{i(\omega t - kx)}$



P_i & P_r ARE 0 OR 180° OUT OF PHASE, DEPENDING UPON THE COEFFICIENTS A_1 & B_1 ;

$$\frac{B_1}{A_1} = \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1}$$

WHEN $\frac{B_1}{A_1} > 0$, P_i & P_r ARE IN PHASE $\Rightarrow \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1} > 0$

WHEN $\frac{B_1}{A_1} < 0$, P_i & P_r ARE 180° OUT OF PHASE $\Rightarrow \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1} < 0$

c_1 & c_2 DESCRIBE PROPAGATION OF INCIDENT AND TRANSMITTED WAVE, AND ARE THUS OF THE SAME SIGN

$$\therefore \frac{\frac{\rho_2 c_2}{\rho_1 c_1} - 1}{\frac{\rho_2 c_2}{\rho_1 c_1} + 1} > 0 \Rightarrow \frac{\rho_2 c_2}{\rho_1 c_1} > 1 \Rightarrow |\rho_2 c_2| > |\rho_1 c_1|$$

FOR 0° PHASE

$|\rho_2 c_2| < |\rho_1 c_1|$

FOR 180° PHASE

VELOCITY BOUNDRY CONDITIONS: $U_L|_{x=0} = U_R|_{x=0}$

YIELD: $\frac{A_1 - B_1}{\rho_1 c_1} = \frac{A_2}{\rho_2 c_2}$

$\frac{A_1 - B_1}{A_2} = \frac{\rho_1 c_1}{\rho_2 c_2}$

NOW, IF $A_1 > 0$, $A_1 > B_1 \Rightarrow A_1 - B_1 > 0$

FOR 0° PHASE 'TWIXT P_i & P_r, $\frac{\rho_1 c_1}{\rho_2 c_2} > 1$

$\Rightarrow \frac{A_1 - B_1}{A_2} > 1$

$\therefore$ ALL THREE WAVES ARE IN PHASE

FOR 180° PHASE 'TWIXT P_i & P_r, $\frac{\rho_1 c_1}{\rho_2 c_2} < 1$

$\Rightarrow \frac{A_1 - B_1}{A_2} < 1$

$\therefore P_t$ & P_R ARE 180° OUT OF PHASE WITH INCIDENT WAVE P_i

?

P_r out of phase with P_i at boundary if $\rho_2 c_2 < \rho_1 c_1$

Also $P_i = \rho_1 c_1 u_i$ but $P_r = -\rho_1 c_1 u_r$ hence if P_i & P_r are in phase u_i & u_r are out of phase by 180° & vice versa !

PROBLEMS

1. Show that any function $z(s)$ where

$$s = ct - (x \cos \phi + y \sin \phi)$$

satisfies the wave equation $c^2\left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right] = \frac{\partial^2 z}{\partial t^2}$

for arbitrary values of ϕ.

(2. If $\phi = \frac{\pi}{6}$ the function $z(ct - x \cos \phi - y \sin \phi)$ represents a disturbance being propagated in a direction which makes an angle of 30° with the +x axes. What is the direction of propagation of the disturbance $z(ct + x \cos \phi - y \sin \phi)$? of the disturbance $z(ct + x \sin \phi - y \cos \phi)$?

✓3. Show that an harmonic solution of the wave equation of the form

$$z(x,y,t) = [d_1 \cos \alpha x + d_2 \sin \alpha x][d_3 \cos \beta y + d_4 \sin \beta y][d_5 \cos \omega t + d_6 \sin \omega t]$$

can be written as

$$z(x,y,t) = A\left[\cos(\alpha x + \beta y - \omega t + \Omega_1) + \cos(\alpha x + \beta y + \omega t + \Omega_2)\right.$$
$$\left. + \cos(\alpha x - \beta y - \omega t + \Omega_3) + \cos(\alpha x - \beta y + \omega t + \Omega_4)\right)$$

where $\beta = \sqrt{(\omega/c)^2 - \alpha^2}$ and $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and A are constants related to d_1, d_2 . d_6. Each of the four quantities in the brackets represents an harmonic wave. Given $\alpha = 1\ m^{-1}$, $\beta = \sqrt{3}\ m^{-1}$ find the direction of propagation of each of these waves.

✓4. Describe the motion of a membrane with a circular boundary vibrating in a characteristic mode for which m = 2, n = 3.

2.4
2.4
96
48
4,76

$\sqrt{14.76}$

(5. For any integer m and arbitrary k, Bessel's equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2}\right)R = 0$$

has a solution $J_m(kr)$. However only for special values of k say k_{m1}, k_{m2}, k_{m3} . . . do these solutions satisfy the condition that $J_m(ka) = 0$. Let $J_m(k_{mn}r)$ and $J_m(k_{mn'}r)$ be two such solutions. Show that

$$\int_0^a J_m(k_{mn}r)\, J_m(k_{mn'}r)\, 2\pi r\, dr = 0 \qquad \text{if } n \neq n'$$

Hint. Use Stoke's theorem

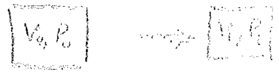
$$\oint \vec{A}\cdot d\vec{l} = \iint_S \text{curl}\,\vec{A}\cdot d\vec{S}$$

with

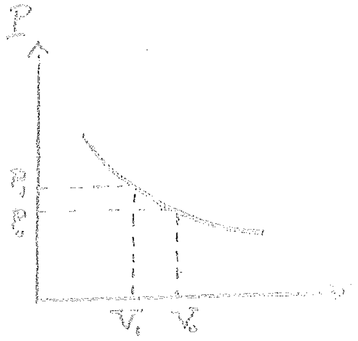
$$\vec{A} = \left[J\,\frac{\partial J'}{\partial r} - J'\,\frac{\partial J}{\partial r}\right]\hat{\phi}$$

where $J = J_m(k_{mn}r)$, $J' = J_m(k_{mn'}r)$ and $\hat{\phi}$ is a unit vector in the ϕ-direction.

11. When a gas expands from a volume V_1 to an indefinitely large volume where the pressure is essentially zero the work done by the gas may be written as $\int_{V_1} PdV$. This then is the potential energy stored in a mass of gas of volume V_1. If an element of the gas occupies a tiny volume V_0 when the pressure is P_0, the potential energy of this element may be taken to be $\int_{V_0} PdV$.



If the pressure changes to some new value P_1, the volume of the element will change to say V_1 and the new value of the potential energy is $\int_{V_1}^{\infty} PdV$. When a sound wave exists in the gas, the pressure at the point where the element is located will vary about some equilibrium value, say P_0. At an instant of time when the pressure is P_1 and the volume of the element is V_1, the additional energy stored in the element over and above that when no wave is present is



$$dE = \int_{V_1}^{\infty} PdV - \int_{V_0}^{\infty} PdV = \int_{V_1}^{V_0} PdV$$

Assuming P varies linearly with V between P_0 and P_1 show that

$$d\mathcal{E} = \left[P_0 + \tfrac{1}{2}\mathcal{P}\right]\left[V_0 - V\right]$$

where $\mathcal{P} = P_1 - P_0$. If P varies harmonically in time, then V will also vary harmonically in time, and if one averages $d\mathcal{E}$ over one cycle (or over any time interval long compared with the time for 1 cycle) only the terms involving the product of $\mathcal{P}$ and V will be different from zero. Thus one may write

$$d\mathcal{E}_{av} = \left\{ \frac{\mathcal{P}}{2}\left[V_0 - V\right] \right\}_{av}$$

Since for any small change in the volume $(V_0 - V)$ one has $\mathcal{P} = -B_a \frac{V - V_0}{V_0}$ one can write

$$d\mathcal{E}_{av} = \left(\frac{\mathcal{P}^2 V_0}{2B_a}\right)_{av}$$

and

$$\frac{d\mathcal{E}_{av}}{V_0} = \left(\frac{\mathcal{P}^2}{2B_a}\right)_{av}$$

The quantity $\frac{dE_{av}}{V_0}$ can be interpreted as the average potential energy per unit volume of the gas. Show that for a plane wave, P = A cos(ωt-kx),

$$\frac{dE_{av}}{V_0} = \frac{A^2}{4B_a} = \frac{A^2}{4\rho c^2}$$

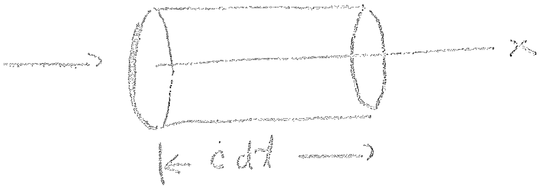
For the same plane wave the average kinetic energy of an element is $\frac{1}{2}(\rho V_0)u_{av}^2$ where u is the particle velocity. Hence ~~and~~ $\frac{1}{2}\rho u_{av}^2$ is the average kinetic energy per unit volume.

The quantity

$$\left(\frac{P^2}{2B_a}\right)_{av} + \frac{1}{2}\rho u_{av}^2$$

is referred to as the energy density at a point. Show that for the plane wave P = A cos (ωt-kx) the energy density is given by $A^2/2\rho c$. If a progressive wave P = A cos (ωt-kx) reaches a surface S at time t, then at time t + dt it will have reached a surface a distance cdt from S, and thus all of the elements in the volume cSdt will have on the average some additional energy over and above that before the wave was present. All this energy passed through surface S in time dt. Show from these considerations that the intensity due to this wave is

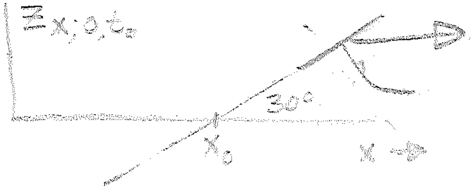
$$I = \frac{A^2}{2\rho c}$$



MARKS

$\frac{4.9}{7}$

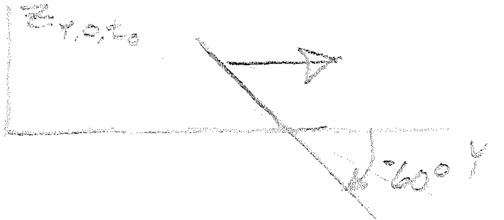
2) $\Xi(ct - x\cos\phi - Y\sin\phi)\ ;\ \phi = \frac{\pi}{6}$

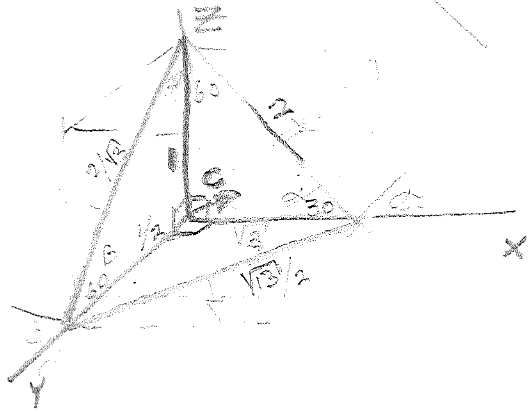


$\Xi = \frac{\delta \Xi}{\delta x}\Big|_{x_0}(x - x_0)$

$Z_{0,Y,t} = \Xi(ct - Y\cos(\phi - \frac{\pi}{2}))$

$= \Xi(ct - Y\cos(-\frac{3\pi}{2}))$





DIRECTION OF C:

$\left(\cos 30^\circ\hat{i}, \cos 60^\circ\hat{j}, 0\hat{k}\right)$

$\left(\frac{\sqrt{3}}{2}\hat{i}, \frac{1}{2}\hat{j}\right)$

$\Xi(ct + x\cos\phi - Y\overset{\sin}{(\phi)}) \Rightarrow$ DIRECTION OF C:

$\left(-\frac{\sqrt{3}}{2}\hat{i}, \frac{1}{2}\hat{j}\right)$ ✓

$\Xi(ct + x\sin\phi - Y\cos\phi) \Rightarrow$ DIRECTION OF C:

$\left(\frac{1}{2}\hat{i}, \frac{\sqrt{3}}{2}\hat{j}\right)$ ✓

3) $z(x,y,t) = [d_1 \cos \alpha x + d_2 \sin \alpha x][d_3 \cos BY + d_4 \sin BY][d_5 \cos \omega t + d_6 \sin \omega t]$

$$= [(d_1^2+d_2^2)(d_3^2+d_4^2)(d_5^2+d_6^2)]^{1/2}$$

$$\cdot \cos(\alpha x + \operatorname{atan}\frac{d_2}{d_1}) \cdot \cos(BY + \operatorname{atan}\frac{d_4}{d_3}) \cdot \cos(\omega t + \operatorname{atan}\frac{d_6}{d_5})$$

$$\Rightarrow A_1 = [(d_1^2+d_2^2)(d_3^2+d_4^2)(d_5^2+d_6^2)]^{1/2}; \Omega_{nm} = \operatorname{atan}\frac{d_n}{d_m}$$

$$z(x,y,t) = A_1 \cos(\alpha x + \Omega_{21}) \cos(BY + \Omega_{43}) \cos(\omega t + \Omega_{65})$$

$$= \frac{A_1}{2}[\cos(\alpha x - BY + \Omega_{21} - \Omega_{43})$$

$$+ \cos(\alpha x + BY + \Omega_{21} + \Omega_{43})] \cos(\omega t + \Omega_{65})$$

$$= \frac{A_1}{4}[\cos(\alpha x - BY - \omega t + \Omega_{21} - \Omega_{43} - \Omega_{65})$$

$$+ \cos(\alpha x - BY + \omega t + \Omega_{21} - \Omega_{43} + \Omega_{65})$$

$$+ \cos(\alpha x + BY - \omega t + \Omega_{21} + \Omega_{43} - \Omega_{65})$$

$$+ \cos(\alpha x + BY + \omega t + \Omega_{21} + \Omega_{43} + \Omega_{65})]$$

$$A = \frac{A_1}{4} \; ; \; \Omega_1 = \Omega_{21} + \Omega_{43} - \Omega_{65}$$

$$\Omega_2 = \Omega_{21} + \Omega_{43} + \Omega_{65}$$

$$\Omega_3 = \Omega_{21} - \Omega_{43} - \Omega_{65}$$

$$\Omega_4 = \Omega_{21} - \Omega_{43} + \Omega_{65}$$

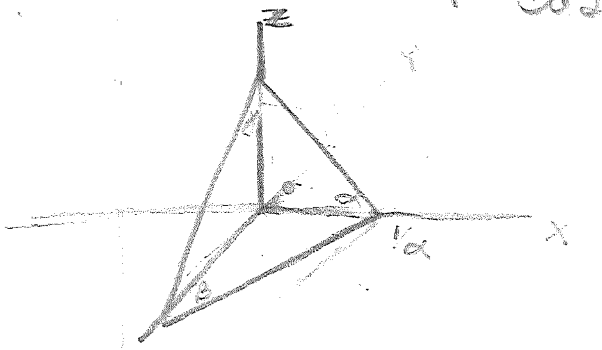
$$\therefore z(x,y,t) = A[\cos(\alpha x + BY - \omega t + \Omega_1)$$

$$+ \cos(\alpha x + BY + \omega t + \Omega_2)$$

$$+ \cos(\alpha x - BY - \omega t + \Omega_3)$$

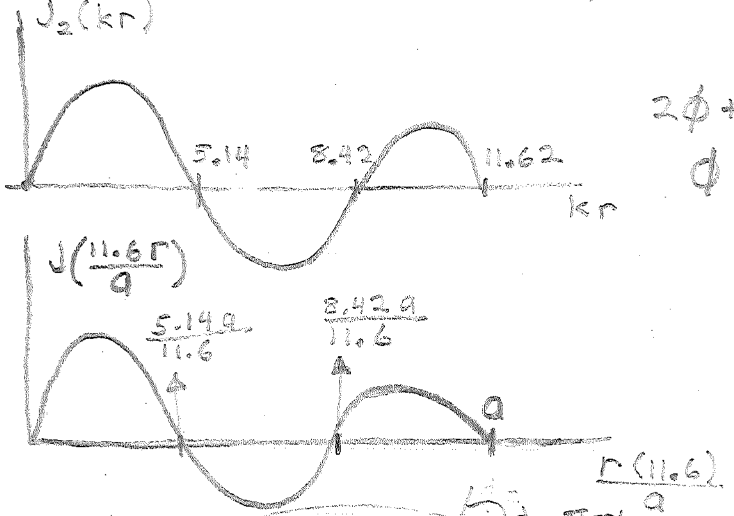
$$+ \cos(\alpha x - BY + \omega t + \Omega_4)]$$

✓



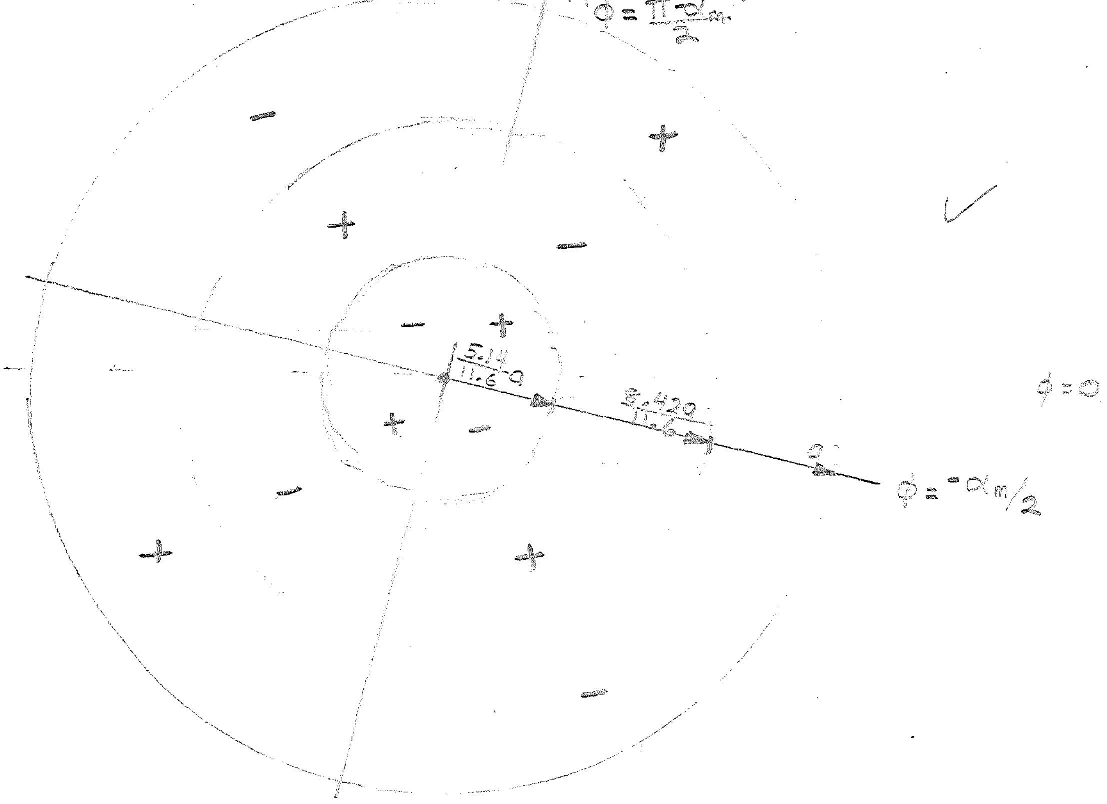
RESPECTIVELY

4) $Z_{23}(r,\phi,t) = C_{23} J_2(kr) [\sin(2\phi + \alpha_m)][\cos(\omega t + \Omega_m)]$



$2\phi + \alpha_m = n\pi$

$\phi = \frac{n\pi - \alpha_m}{2}$



✓

4-9) a) $T = (2 \times 10^{8})(2 \times 10^{-5}) = 4000 \frac{nt}{m}$ ✓ .2

b) $J_0(k_{01} r) = 2.40$

$k_{01} = \frac{2.40}{a} = \frac{2.40}{3 \times 10^{-2}} = 80 m^{-1}$

$\Rightarrow \omega = k_{01} C$

$= 80 C$

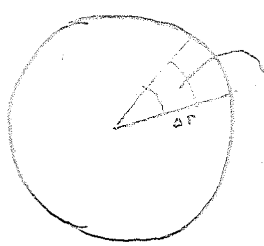
$C = \sqrt{T/\sigma} = \sqrt{\frac{80}{33/13}} =$?

yes?

(61-17

Lump these together

4-5) $Z_0(r,\phi,t) = C_0 J_0(kr)\left[\sin\alpha_m\right]\left[\cos(\omega t + \Omega_m)\right]$



$dA = r\,dr\,d\phi$

@ $\cos(\omega t + \Omega_m) = 0$; ENERGY IS ALL KINETIC $= \frac{1}{2}mv^2$

$\Rightarrow \omega t + \Omega_m = \frac{\pi(2p+1)}{2} \Rightarrow t_p = \frac{1}{\omega}\left[\frac{(2p+1)\pi}{2} - \Omega_m\right] \quad = \frac{1}{2}m\left(\frac{dz}{dt}\right)^2$

ON dA: $\quad p = 0, 1, 2, \ldots$

$$\Delta E = \frac{1}{2}(\sigma r\,dr\,d\phi)\left(\frac{\delta z}{\delta t}\Big|_{r,\phi,t_n}\right)^2$$

$$\frac{dz}{dt} = C_0\omega J_0(kr)\sin\alpha_0 \sin(\omega t + \Omega_m)$$

$$\frac{\delta z}{\delta t}\Big|_{t_p} = C_0\omega J_0(kr)\sin\alpha_0 \quad (-1)^p$$

$$\left(\frac{\delta z}{\delta t}\Big|_{t_p}\right)^2 = C_0^2\omega^2 J_0^2(kr)\sin^2\alpha_0$$

$$\Rightarrow \Delta E = \frac{1}{2}\sigma r C^2\omega^2 J_0^2(kr)\sin^2\alpha_0\,dr\,d\phi$$

$$k = \frac{2.40}{a}$$

$$\Rightarrow E = \frac{1}{2}\sigma C_0^2\omega^2\sin^2\alpha_0\int_0^{2\pi}\int_0^a r J_0^2\left(\frac{2.4r}{a}\right)dr\,d\phi$$

$$= \frac{a}{2.4}\frac{1}{2}\sigma C_0^2\omega^2\sin^2\alpha_0\int_0^{2\pi}\int_0^a x J_0^2(x)\,dr\,d\phi$$

$$\Rightarrow x = \frac{2.4r}{a} \Rightarrow r = \frac{a}{2.4}x;\ dr = \frac{a}{2.4}dx;\ r=a \Rightarrow x = 2.4$$

$$= \left(\frac{a}{2.4}\right)^2\frac{1}{2}\sigma A^2\omega^2\int_0^{2\pi}\int_0^{2.4} x J_0^2(x)\,dx\,d\phi$$

$$= \left(\frac{a}{2.4}\right)^2\frac{1}{2}\sigma A_1^2\omega^2 2\pi \quad \frac{x^2}{2}\left[J_0^2(x) + J_1^2(x)\right]_0^{2.4}$$

$$= \frac{1}{(2.4)^2}a^2\sigma A_1^2\omega^2\pi\left[\frac{(2.4)^2}{2}(0.5)^2\right]$$

$$= 0.135\,\pi a^2\sigma\omega^2 A_1^2 \checkmark$$

5) $\frac{d^2J}{dr^2} + \frac{1}{r}\frac{dJ}{dr} + \left(k^2 - \frac{m}{r^2}\right)J = \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ}{dr}\right) + \left(k^2 - \frac{m^2}{r^2}\right)J\right] = 0$

$\Rightarrow \quad J'\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ}{dr}\right) + \left(k^2 - \frac{m^2}{r^2}\right)J\right] = 0$

$J\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ'}{dr}\right) + \left(k'^2 - \frac{m^2}{r^2}\right)J'\right] = 0$

SUBTRACTING: IF $J' \neq J$ $(\Rightarrow n \neq n')$

$\Rightarrow J'\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ}{dr}\right) + \left(k^2 - \frac{m^2}{r^2}\right)J\right] = J\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ'}{dr}\right) + \left(k^2 - \frac{m^2}{r^2}\right)J'\right]$

~~$J'JK + J'\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ}{dr}\right) - \frac{m^2}{r^2}J\right] = J'JK + J\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dJ'}{dr}\right) - \frac{m^2}{r^2}J'\right]$~~

$J'Jk^2r + J'\left[\frac{d}{dr}\left(r\frac{dJ}{dr}\right) - \frac{m^2}{r}J\right] = J'Jk'^2r + J\left[\frac{d}{dr}\left(r\frac{dJ'}{dr}\right) - \frac{m^2}{r}J'\right]$

$[k^2 - k'^2]\, JJ'r = J\left[\frac{d}{dr}\left(r\frac{dJ'}{dr}\right) - \frac{m^2}{r}J'\right] - J'\left[\frac{d}{dr}\left(r\frac{dJ}{dr}\right) - \frac{m^2}{r}J\right]$

$= J\frac{d}{dr}\left(r\frac{dJ'}{dr}\right) - J'\frac{d}{dr}\left(r\frac{dJ}{dr}\right)$

$[k^2 - k'^2]\int_0^a JJ'r\,dr = \int_0^a \left[J\frac{d}{dr}\left(r\frac{dJ'}{dr}\right) - J'\frac{d}{dr}\left(r\frac{dJ}{dr}\right)\right]dr$

$= \int_0^a J\frac{d}{dr}\left(r\frac{dJ'}{dr}\right)dr - \int_0^a J'\frac{d}{dr}\left(r\frac{dJ}{dr}\right)dr$

$u = J \qquad dv = \frac{d}{dr}\left(r\frac{dJ'}{dr}\right)dr$

$du = \frac{dJ}{dr}dr \qquad v = r\frac{dJ'}{dr}$

$[k^2 - k'^2]^a\int_0^a rJJ'dr = \left[rJ\frac{dJ'}{dr} - rJ'\frac{dJ}{dr}\right]_0^a$

$- \int_0^a \left[r\frac{dJ}{dr}\frac{dJ'}{dr} - r\frac{dJ'}{dr}\frac{dJ}{dr}\right]dr$

$J(ka) = J'(k'a) = 0$

$\Rightarrow [k^2 - k'^2]\int_0^a rJJ'dr = -\int_0^a r\left[\frac{dJ}{dr}\frac{dJ'}{dr} - \frac{dJ'}{dr}\frac{dJ}{dr}\right]dr$

$= 0$ ✓

$\Rightarrow \int_0^a 2\pi r\, J_m(k_{mn}r)\, J_m(k_{mn'}r)\, dr = 0$

$J = J_m(k_{mn}r) \;;\; J' = J_m(k_{mn'}r)$

$k = k_{mn} \;;\; k' = k_{mn'}$

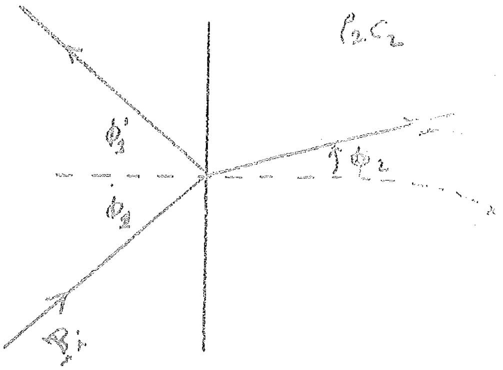
④ An acoustic pressure wave

$$\underset{\sim}{\mathcal{P}} = \underset{\sim}{A_1} e^{i[\omega t - k_1(x\cos\phi_1 + y\sin\phi_1)]}$$

is incident at an angle ϕ_1 on the boundary separating two media. Assuming the law of reflection and refraction hold

a) write down suitable expressions for the reflected and refracted waves.

b) Write down the boundary conditions that must exist at the interface.



c) in terms of the amplitudes of the separate waves, the angles ϕ_1, ϕ_2, and the characteristic impedances $\rho_1 c_1$, and $\rho_2 c_2$ write down an energy balance equation that must hold at the boundary. (30)

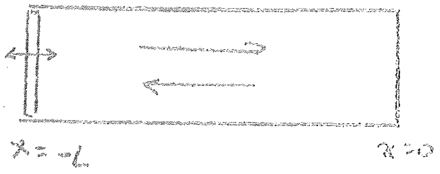
⑤ A piston vibrating harmonically at one end of a closed pipe of cross section S and length L sets up waves in the pipe such that the acoustic pressure at any point is given by

$$\underset{\sim}{\mathcal{P}} = \underset{\sim}{A} e^{i(\omega t - kx)} + \underset{\sim}{B} e^{i(\omega t + kx)}$$

a) What is the ratio of $\underset{\sim}{B}$ to $\underset{\sim}{A}$ determined from the boundary condition at $x = 0$? (5)

b) Calculate the acoustical impedance at the point where the piston is located. (5)

c) The acoustical impedance at the point where the piston is located may be called the input impedance. The resonant frequencies of such a tube may be defined as those frequencies for which the imaginary part of the input impedance vanishes. What resonant frequencies does this definition predict for this closed tube? (20)



$\frac{3}{5}$

7

7-4) $W = 10^{-2}$ WATTS ; $f = 400$ HZ ; $r = 0.5$

a) $W = 4\pi r^2 I$ ✓ 3.19×10^{-3}

$$\Rightarrow I = \frac{W}{4\pi r^2} = \frac{10^{-2}}{4\pi(0.25)} = \frac{10^{-2}}{\pi} = \boxed{0.319} \ \frac{\text{WATTS}}{m^2}$$

b) $I = \frac{p^2}{2\rho_0 c}$

$$\Rightarrow p = \sqrt{2 I \rho_0 c} \quad ✓$$

@ 20° C, $c = 343 \frac{m}{sec}$ IN AIR

$\quad \& \ \rho_0 = 1.21 \frac{kg}{m^3}$

$$\Rightarrow p = \sqrt{2(0.319)(343)(1.21) \times 10^{-3}}$$

$$= \sqrt{2650 \times \boxed{10^2}\, 10^{-1}}$$

$$= \boxed{51.5 \ \boxed{Nt}} \qquad 1.62 \ \frac{nt}{m}$$

c) $U = \frac{A}{\rho_0 c k} \frac{1}{r^2} \sqrt{1 + k^2 r^2}$

$$= \frac{p}{\rho_0 c k r} \sqrt{1 + k^2 r^2}$$

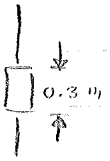
$$= \frac{p}{\rho_0 \omega r} \sqrt{1 + \left(\frac{\omega}{c}\right)^2 r^2}$$

$$= \frac{p}{\rho_0 2\pi f r} \sqrt{1 + \left(\frac{\omega}{c}\right)^2 (r^2)}$$

$$= \frac{\boxed{51.5} \ \ 1.62}{(1.21)(2\pi)(400)(0.5)} \left[1 + \left[\frac{2\pi(400)}{343(0.5)}\right]^2\right]^{\frac{1}{2}}$$

$$= 4.05 \times 10^{-3} \ \frac{m}{sec}$$

7-13)



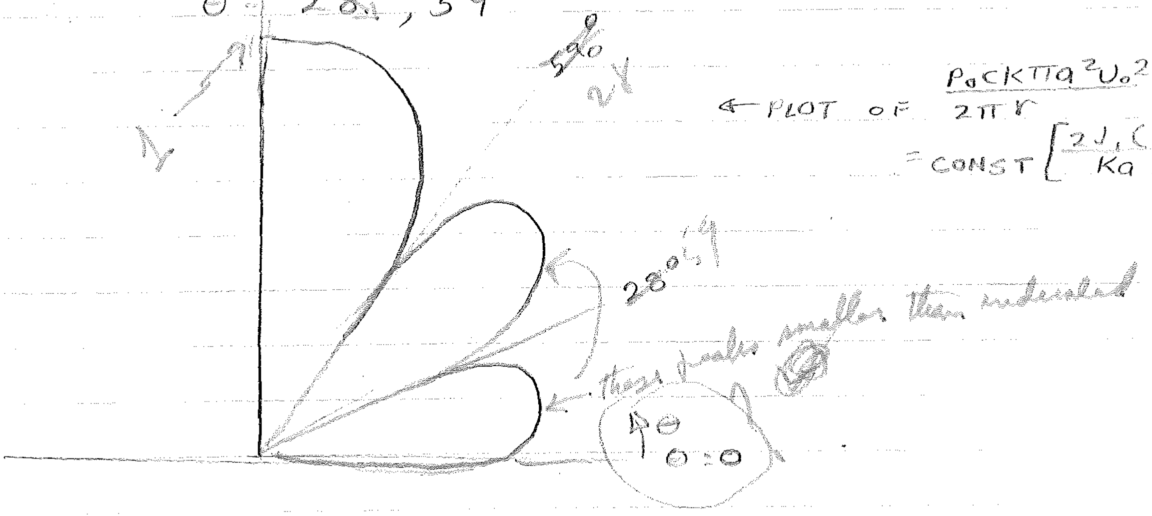
$\omega = 2\pi \times 3000$

$\frac{2\pi}{\lambda} a = ak = a\frac{\omega}{c} = \frac{(0.15)(2\pi)(3000)}{343} = 8.24$ ✓

$\Rightarrow \sin\theta = \frac{3.83}{8.24}, \frac{7.02}{8.24}, \left(\frac{10.17}{8.24} \dots \rightarrow \text{NO SOLUTIONS!}\right)$

$= 0.465, 0.853$

$\theta = 28°, 59°$



$\leftarrow$ PLOT OF $\frac{\rho_0 c k \pi a^2 U_0}{2\pi r} \frac{2 J_1(ka\sin\theta)}{ka\sin\theta}$
$= \text{CONST}\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right)$

$ka = 8.24$

@ WALL, $\theta = 0$

$\text{CONST}\left[\underbrace{\frac{2J_1(ka\sin\theta)}{ka\sin\theta}}_{\Rightarrow 2}\right] = 2\text{CONST}$

ON AXIS $\theta = \frac{\pi}{2}$

$\text{CONST}\left[\frac{2J_1(ka)}{ka}\right] = 2\,\text{CONST}$

UH OH!

On wall $\sin\theta = \frac{\pi}{2}$

Find ka such that

$\frac{2J_1(ka)}{ka} = \frac{1}{2}$

$\Rightarrow 4J_1(ka) = (ka)$

From tables ka = 2.2 will satisfy above eq.

$\frac{2\pi f}{c} a = 2.2$

$f = 800$ Hz

7-8) $u_s = U_0 e^{j\omega t}$

@ $r = a$, THE SPHERE'S MOTION IS THAT OF THE ACOUSTIC WAVE:

$$\Rightarrow \frac{A}{a z_a} e^{-jka} = U_0$$

OR $\underline{A} = U_0 a z_a e^{-jka}$ ✓

WHERE z_a = SPECIFIC ACOUSTIC IMPEDANCE OF A SPHERICAL WAVE

$$z_a = \rho_0 c \frac{kr}{1+(kr)^2}\left[kr + j\right]\Big|_{r=a} = \rho c \frac{ka}{1+(ka)^2}\left[ka + j\right]$$

$$\Rightarrow A = U_0 \frac{aka}{1+(ka)^2}\left[ka+j\right] e^{-jka}$$

P = PRESSURE AMBLITUDE

$$P = \left|\frac{A}{r}\right| = \frac{1}{r}\left|\frac{akU_0}{1+(ka)^2}\left[ka+j\right] e^{-jka}\right|$$

$$= \left|\frac{akU_0}{1+(kr)^2}\sqrt{(kr)^2+1}\, e^{-j(ka - \operatorname{atan} 1/kr)}\right|$$

$$= \frac{\check{a}kU_0}{\sqrt{1+(ka)^2}\, r} \ni k = \omega/c$$

I = SPHERICAL WAVE INTENSITY

$$= \frac{P^2}{2\rho_0 c}$$

$$= \frac{1}{2\rho_0 c}\left[\frac{(\check{a}kU_0)^2}{1+(ka)^2}\right]\frac{1}{r^2} \ni k = \omega/c$$

$$= \frac{1}{(ka)^2}(ka\cos\theta)\left[\frac{-ka\sin\theta}{\sqrt{(ka)^2-(ka)^2\sin^2\theta}}\right]\left[2J_1(v)-vJ_2(v)\right]$$
$$+\left[(ka)^2-(ka)^2\sin^2\theta\right]^{1/2}\left[\frac{2J_1(v)}{v}-2J_2(v)+vJ_3(v)\right]$$

$$= \frac{-1}{ka}\cos\theta\,\frac{ka\sin\theta}{ka\sqrt{1-\sin^2\theta}}\left[2J_1(v)-vJ_2(v)\right]$$
$$+ka\left[1-\sin^2\theta\right]^{1/2}\left[\frac{2J_1(v)}{v}-2J_2(v)+vJ_3(v)\right]$$

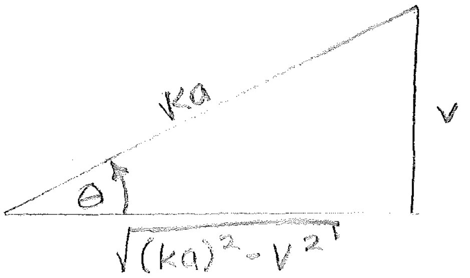
$$= \frac{-1}{ka}\cos\theta\tan\theta\left[2J_1(v)-vJ_2(v)\right]$$
$$+ka\cos\theta\left[\frac{2J_1(v)}{v}-2J_2(v)+vJ_3(v)\right]$$

$$= \frac{-1}{ka}\sin\theta\left[2J_1(v)-vJ_2(v)\right]$$
$$+ka\cos\theta\left[\frac{2J_1(v)}{v}-2J_2(v)+vJ_3(v)\right.$$

$$= \frac{-1}{ka}\sin\theta\left[2J_1(ka\sin\theta)-ka\sin\theta\,J_2(ka\sin\theta)\right]$$
$$+ka\cos\theta\left[\frac{1}{ka\sin\theta}J_1(ka\sin\theta)-2J_2(ka\sin\theta)\right.$$
$$\left.+ka\sin\theta\,J_3(ka\sin\theta)\right]$$

$$= \frac{-2\sin\theta}{ka}J_1(ka\sin\theta)+\sin^2\theta\,J_2(ka\sin\theta)$$
$$+\cot\theta\,J_1(ka\sin\theta)-2ka\cos\theta\,J_2(ka\sin\theta)$$
$$+(ka)^2\sin\theta\cos\theta\,J_3(ka\sin\theta)$$

$$= \left[\cot\theta-\frac{2\sin\theta}{ka}\right]J_1(ka\sin\theta)$$
$$+\left[\sin^2\theta-2ka\cos\theta\right]J_2(ka\sin\theta)$$
$$+\left[(ka)^2\sin\theta\cos\theta\right]J_3(ka\sin\theta)$$



MARKS

$$\rho(r,\theta,t)=\frac{A_1}{r}e^{i(\omega t-kr)}\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]\Rightarrow\frac{\delta\rho}{\delta\phi}=\frac{\delta^2\rho}{\delta\phi^2}=0$$

$$\frac{\delta^2\rho}{\delta t^2}=\frac{-\omega^2A_1}{r}e^{i(\omega t-kr)}\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right];\ \frac{\delta^2\rho}{\delta r^2}=-k^2\rho;\ \frac{\delta\rho}{\delta r}=-ik\rho$$

$$c^2\left[\frac{\delta^2\rho}{\delta r^2}+\frac{2}{r}\frac{\delta\rho}{\delta r}+\frac{1}{r^2}\frac{\delta^2\rho}{\delta\theta^2}+\cancel{\frac{\cos\theta}{\sin\theta}\frac{\delta\rho}{\delta\phi}}+\cancel{\frac{1}{r^2\sin^2\theta}\frac{\delta^2\rho}{\delta\phi^2}}\right]=\frac{\delta^2\rho}{\delta t^2}$$

$$\frac{\delta\rho(r,\phi,t)}{\delta\theta}=\frac{2A_1}{r}e^{i(\omega t-kr)}\frac{d}{d\theta}\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]$$

LET $V=ka\sin\theta\Rightarrow\frac{dV}{d\theta}=ka\cos\theta$

$$\Rightarrow\frac{d}{d\theta}\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]=\frac{dV}{d\theta}\frac{d}{dV}\left[\frac{J_1(V)}{V}\right]$$

$$=\frac{dV}{d\theta}\left[\frac{V\frac{dJ_1(V)}{dV}+J_1(V)}{V^2}\right]$$

$$=\frac{dV}{d\theta}\left[\frac{\{J_1(V)-VJ_2(V)\}+J_1(V)}{V^2}\right]$$

$$=ka\cos\theta\left[\frac{2J_1(ka\sin\theta)-ka\sin\theta\,J_2(ka\sin\theta)}{(ka\sin\theta)^2}\right]$$

$$=\frac{2J_1(ka\sin\theta)-ka\sin\theta\,J_2(ka\sin\theta)}{ka\tan\theta}$$

$$\frac{d^2}{d\theta^2}\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]=\frac{d}{d\theta}\left[\frac{2J_1(ka\sin\theta)-ka\sin\theta\,J_2(ka\sin\theta)}{ka\tan\theta}\right]$$

$$=\frac{dV}{d\theta}\frac{d}{dV}\left[\frac{2J_1(V)-VJ_2(V)}{(ka)^2/\sqrt{-V^2+(ka)^2}}\right]$$

$$=\frac{1}{(ka)^2}\frac{dV}{d\theta}\frac{d}{dV}\left[(-V^2+k^2a^2)^{\frac{1}{2}}\{2J_1(V)-VJ_2(V)\}\right]$$

$$=\frac{1}{(ka)^2}\frac{dV}{d\theta}\left[\frac{d}{dV}(-V^2+k^2a^2)^{\frac{1}{2}}\{2J_1(V)-VJ_2(V)\}\right.$$

$$\left.+(-V^2+k^2a^2)^{\frac{1}{2}}\frac{d}{dV}\{2J_1(V)-VJ_2(V)\}\right]$$

$$=\frac{1}{(ka)^2}\frac{dV}{d\theta}(-V)(-V^2+k^2a^2)^{-\frac{1}{2}}\{2J_1(V)-VJ_2(V)\}$$

$$+(-V^2+k^2a^2)^{\frac{1}{2}}\left[\frac{2J_1(V)}{V}-2J_2(V)+VJ_3(V)\right]$$

$$
\begin{aligned}
= & \left\{ \frac{\sqrt{(ka)^2 - v^2}}{v} - \frac{2v}{(ka)^2} \right\} J_1(v) \\
& + \left\{ \left(\frac{v}{ka}\right)^2 - 2\sqrt{(ka)^2 - v^2} \right\} J_2(v) \\
& + \left[(ka)^2 \frac{kva}{(ka)^2 - v^2} \right] J_3(v)
\end{aligned}
$$

OH WELL

ACOUSTICS TEST

1. If the external forces exerted on a rectangular block are all normal to the surfaces of the block then the strains resulting from changes in the external forces are given by the following stress-strain relations:

$$\epsilon_{xx} = \frac{1}{Y} S_{xx} - \frac{\sigma}{Y} S_{yy} - \frac{\sigma}{Y} S_{zz}$$

$$\epsilon_{yy} = -\frac{\sigma}{Y} S_{xx} + \frac{1}{Y} S_{yy} - \frac{\sigma}{Y} S_{zz}$$

$$\epsilon_{zz} = -\frac{\sigma}{Y} S_{xx} - \frac{\sigma}{Y} S_{yy} + \frac{1}{Y} S_{zz}$$

where Y and σ are elastic constants characteristic of the material from which the block is made.

a) What are the names of σ and Y and what units would they have in the MKS system? (7)

b) If a block with elastic constants Y & σ has a length ℓ_0 when subjected to a hydrostatic pressure P_0, determine the change in the length that would occur if the pressure were changed to P_1. (7)

c) How is the shear modulus defined and what units would it have in the MKS system? (7)

2. The real part of

$$\underset{\sim}{x}_1 = 2 e^{i(\omega t - \theta)}$$

as well as the real part of

$$\underset{\sim}{x}_2 = 3 \underset{\sim}{x}_1 - 4 i \underset{\sim}{x}_1$$

represent simple harmonic motions.

a) What is the phase difference between $\underset{\sim}{x}_2$ and $\underset{\sim}{x}_1$ and which leads? (7)

b) What is the amplitude of the simple harmonic motion which the real part of $\underset{\sim}{x}_2$ represents?

3. The equation of motion for a driven harmonic oscillator is

(i) $$m\ddot{x} + R\dot{x} + Kx = F_0 \cos\omega t$$

and its steady state solution is the real part of

(ii) $$\underset{\sim}{x} = -i \frac{F_0 e^{i\omega t}}{\omega \underset{\sim}{Z}_m} \quad \text{where} \quad \underset{\sim}{Z}_m = R + i(\omega m - K/\omega)$$

a) Write down an integral which would give the <u>average rate</u> that energy is supplied to the system by the driving force. It is unnecessary to evaluate the integral but define any symbol used that is not given in (i) and (ii) above. (7)

b) How is the Q of such a mechanical system defined? (7)

c) How is the mechanical impedance of such a system <u>defined</u>? (7)

d) What is meant by a mass controlled oscillator? (7)

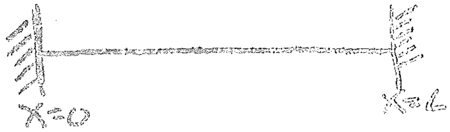
4. The wave equation for waves on strings is

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \qquad c = \sqrt{T/\rho}$$

It is readily shown that the function

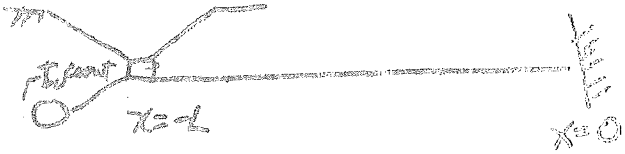
$$y(x,t) = \left[A \cos\tfrac{\omega}{c}x + B \sin\tfrac{\omega}{c}x\right]\cos\omega t + \left[C \cos\tfrac{\omega}{c}x + D \sin\tfrac{\omega}{c}x\right]\sin\omega t$$

is a solution of the wave equation for every positive value of ω and completely arbitrary values of A, B, C, D. Show that if a string of length L is fastened at each end that it is possible to have a solution of the above form which satisfies both boundary conditions provided some of the arbitrary constants are given special values and ω is restricted to certain values. Indicate the special values of the constants and ω.



(16)

5. A string is fastened at one end and driven at the other end by a harmonic oscillator at a frequency ω. Assume a steady state has been achieved. If the velocity of waves on the string is c and the mass per unit length of the string is ρ, find in terms of ρ, c, ω and t (time)



(a) the force the string exerts on the driver at any time t. (7)

(b) The impedance Z at a general point of the string. (7)

(c) The impedance at the driving point of the string. (7)

BOB MARKS

1)a) Y IS YOUNGS MODULUS ; DIMENSIONS OF $\frac{\text{AREA}}{\text{FORCE}} = \boxed{\frac{\text{METER}^2}{\text{NEWTON}}}$ ✓

$\left\{ \epsilon_{zz} = \frac{1}{Y}\frac{F}{A} \right)$ ~~σ IS YOUNG'S MODULUS~~

σ IS POISSON'S RATIO, AND IS DIMENSIONLESS ✓

5

b) $S_{xx} = S_{yy} = S_{zz} = -\Delta p$ ✓

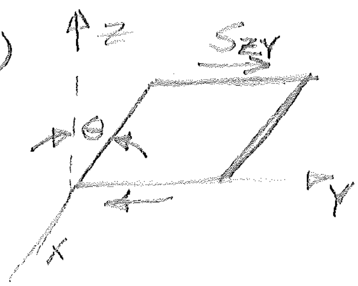
$$\Rightarrow \epsilon_{xx} = -\left(\frac{1}{Y} - \frac{2\sigma}{Y}\right)\Delta p$$

$$\frac{l - l_0}{l_0} = \frac{(2\sigma - 1)}{Y}\Delta p$$

$$\frac{\Delta l}{l_0} \Rightarrow \Delta l = \frac{l_0(2\sigma - 1)}{Y}(P_1^0 - P_0)$$ ✓

7

c)



$G\theta = S_{ZY}$

$G = \frac{S_{ZY}}{\theta}$, AND HAS MKS UNITS OF $\frac{\text{NEWTONS}}{\text{METER}^2\ \text{RADIANS}}$ ✓

7

I	a	5	II	a 7	III	a 4	IV	14
	b	7		b 7		b 7		
	c	7				c 0		
						d 7		

$\boxed{83}$

2)a) ~~$X_1 = 2e^{i(wt-\theta)}$~~

~~$X_2 = 3X_1 + 4iX_2 = 5e^{i \,\mathrm{atan}\frac{4X_2}{3X_1}}$~~

~~$= 6e^{i(wt-\theta)} + 4iX_2$~~

$X_1 = 2e^{i(wt-\theta)}$

$X_2 = 6e^{i(wt-\theta)} - i8e^{i(wt-\theta)}$

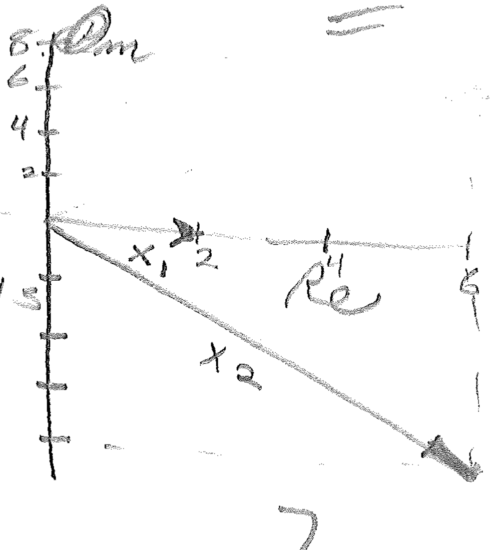
$= 6e^{i(wt+\theta)} - (e^{\pi i})8e^{i(wt-\theta)}$

$= (6 + 8e^{i\pi})e^{i(wt-\theta)}$

$\mathrm{atan}\frac{8}{6} = \mathrm{atan}\frac{4}{3} = \mathrm{atan}\,1.\overset{33}{\boxed{25}}$

7

X₁ LEADS X₂ BY atan 1.25 RADIANS ✓



b) ~~$Re X_2 = Re(6 - i8)\cos wt - \theta$~~

$|X_2| = \sqrt{6^2 + 8^2}$

$= \sqrt{36 + 64} = 10 \Rightarrow$ AMPLITUDE ✓

7

$\int F_0 \cos\omega t \; \frac{F_0 \cos(\omega t - \theta)}{Z_m} \, dt$

3) a) ~~$P_{AVE} = [illegible] \; R^2 |\dot{x}|^2$~~

~~$= R^2 \left[\frac{F_0 e^{i\omega t}}{Z_m} \right.$~~

$\boxed{\tau = \frac{2\pi}{\omega}}$

~~$\dot{x} = \frac{F_0 e^{i\omega t}}{\omega Z_m} \Rightarrow |\dot{x}| = \frac{F_0}{\omega |Z_m|}$~~

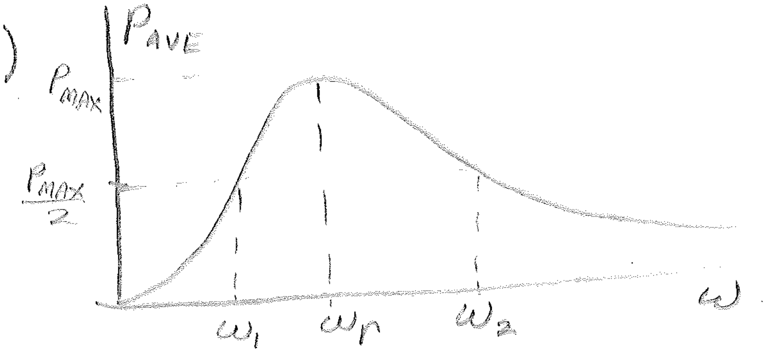
~~$Re(\dot{x}) =$~~

4

$$P_{AVE} = \frac{1}{\tau} \int_0^\tau R \, (\dot{x})^2 \, dt$$

$$= \frac{1}{\tau} \int_0^\tau R^2 \, Re \left[\frac{F_0 e^{i\omega t}}{\omega Z_m} \right] dt$$

b)



$Q = \frac{\omega_r}{\omega_2 - \omega_1}$ ✓ 7

c) $Z = \boxed{\frac{T \, \delta Y / \delta x}{\delta Y / \delta t}}$ 0

d) $\omega M >> \frac{K}{\omega}$

$\omega m >> R$ ✓ 7

$$Y(0,t) = 0 = A\cos\omega t + C\sin\omega t$$

$$Y(0,L) = 0 = \left(A\cos\frac{\omega}{c}L - B\sin\frac{\omega}{c}L\right)\cos\omega t + \left[C\cos\frac{\omega L}{c} - D\sin\frac{\omega L}{c}\right]\sin\omega t$$

$$\Rightarrow \cos\frac{\omega}{c}L\left[A\cos\omega t \overset{0}{\cancel{+A}} C\sin\omega t\right]$$

$$-\sin\frac{\omega}{c}L\left(B\cos\omega t + D\sin\omega t\right]$$

$$\Rightarrow \sin\frac{\omega}{c}L\left(B\cos\omega t + D\sin\omega t\right) = 0$$

~~$\Rightarrow B = D = 0$~~

$$\Rightarrow \boxed{\omega = \frac{n\pi c}{L}} \quad ✓$$

$$Y(0,t) = 0 = A\cos\frac{n\pi c}{L}t + C\sin\frac{n\pi c}{L}t$$

14

$$\therefore A = 0 = C$$

5)

a) $F = -T \left. \frac{\delta Y}{\delta X} \right|_{x=-L}$

$Y_n(x,t) = A \sin \frac{n\pi}{L} x \cos \frac{n\pi c}{L} t$

$\left. \frac{dY_n}{dx} \right|_{-L} = + \frac{A n\pi}{L} \cos \frac{n\pi}{L} x \sin \frac{n\pi c}{L} t$

$\Rightarrow F = \frac{TAn\pi}{L} \sin \frac{n\pi c}{L} t$

b) $Z_s = \frac{T\, dY/dx}{\delta Y/\delta t}$

$= \frac{+T \frac{\cancel{DnA}}{L} \cos \frac{n\pi}{L} x \cos \frac{n\pi c}{L} t}{+\cancel{Dn\pi cA} \sin \frac{n\pi}{L} x \sin \frac{n\pi c}{L} t}$

$= T\, c \cot \frac{n\pi}{L} x \cot \frac{n\pi c}{L} t$

e) $Z_{dp} = Z_s$ ~~$\big|_{x=-L} = T\, C \cot(-n\pi)$~~

GENERAL INFORMATION

Eigen functions for membrane with rectangular boundary

$$z_{mn} = A_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \cos (\omega_{mn} t + \omega_{mn})$$

$$\omega_{mn} = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \begin{array}{l} m = 1, 2, 3, \ldots\ldots \\ n = 1, 2, 3, \ldots\ldots \end{array}$$

ACOUSTICS TEST

October 23, 1972

1. What factors determine the speed with which longitudinal waves are propagated in a thin rod? What factors determine the speed with which torsional waves are propagated in a thin rod? The speed with which transverse waves are propagated? (20)

2. The wave equation for <u>longitudinal</u> waves in a rod is

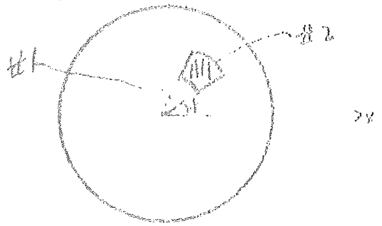
$$c^2 \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial^2 \xi}{\partial t^2}$$

and a harmonic solution is

$$\xi(x,t) = (a_1 \cos \frac{\omega}{c} x + b_1 \sin \frac{\omega}{c} x)(a_2 \cos \omega t + b_2 \sin \omega t)$$

Determine the values of ω for which such a function will satisfy the boundary conditions if the rod is clamped at $x=0$ and free at $x=L$. (25)

3. The motion of a membrane with a circular boundary is described by some function $z(r,\phi,t)$. Consider a small element of the membrane of area $r\Delta\phi\Delta r$ and write down for some general time t the net vertical component of the two forces acting on sides labelled #1 and #2 in the figure. Assume the tension in the membrane is T. (20)

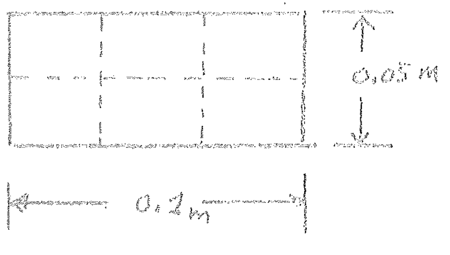


4. An harmonic solution of the wave equation for waves in a membrane is in polar coordinates

$$z(r,\phi,t) = J_m(kr)\left[A \cos m\phi + B \sin m\phi\right]\left[a \cos \omega t + b \sin \omega t\right]$$

Explain why m must be an integer. (10)

5. A membrane is clamped in a rectangular frame 0.1 m by 0.05 m and some salt is spread over its surface. When the membrane is set into vibration by a loud speaker driven at a frequency of 500 CPS, the salt forms a pattern shown by the dotted lines. If the membrane has a surface density of $\sigma = 1$ kg/m^2 what is the tension in the membrane? (25)



BOB MARKS

① LONGITUDINAL

$$c^2 \frac{\delta^2 \xi}{\delta x^2} = \frac{\delta^2 \xi}{\delta t^2} \Rightarrow c = \sqrt{Y/\rho}$$ ✓ Y: YOUNG'S MODULUS

ρ = VOLUME MASS DENSITY 6

TORSIONAL

$$c^2 \frac{\delta^2 \psi}{\delta x^2} = \frac{\delta^2 \psi}{\delta t^2} \Rightarrow c = \sqrt{G/\rho}$$ ✓ G = SHEAR MODULUS

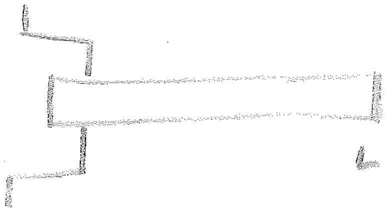
ρ = VOLUME MASS DENSITY

TRANSVERSE

$$-(cI_0)^2 \frac{\delta^4 Y}{\delta x^4} = \frac{\delta^2 Y}{\delta t^2} \Rightarrow c = \sqrt{Y/\rho} \quad I_0 = \frac{h}{\sqrt{12}}$$

THE VELOCITY OF A TRANSVERSE IS DEPENDENT OF FREQUENCY OF WAVE. ✓ and also Y, ρ & I_0 4

② x = 0 CLAMPED $\Rightarrow \xi(0,t) = 0$



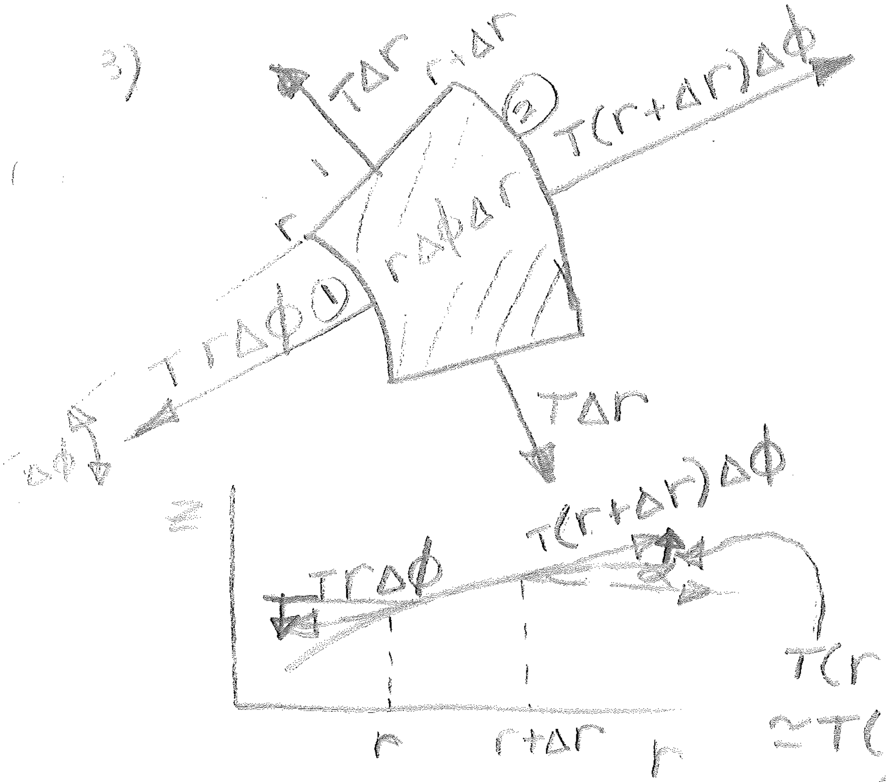
$$x = L \text{ FREE} \Rightarrow \left.\frac{\delta \xi}{\delta x}\right|_{L,t} = 0 \qquad \left(\epsilon_{xx} = \frac{F_x}{YA} = 0\right)$$
$$= \frac{\delta \xi}{\delta x}$$

$$\xi(0,t) = (a_1)(a_2 \cos wt + b_2 \sin wt)$$
$$\Rightarrow a_1 = 0$$
$$\Rightarrow \xi(x,t) = b_1 \sin \frac{w}{c} x (a_2 \cos wt + b_2 \sin wt)$$
$$\frac{\delta \xi}{\delta x} = +\left(\frac{w}{c}\right) b_1 \cos \frac{w}{c} x (a_2 \cos wt + b_2 \sin wt)$$
$$\left.\frac{\delta \xi}{\delta x}\right|_{L,t} = 0 \Rightarrow \cos \frac{w}{c} L = 0$$
$$\frac{w}{c} L = \frac{(2n+1)\pi}{2} \Rightarrow n \text{ IS AN INTEGER}$$
$$\omega_n = \frac{c}{2L}(2n+1)\pi$$ ✓ 25

I 16
II 25
III 20
IV 10
V 25

96

3)



$$T(r+\Delta r)\Delta\phi \sin\alpha$$

$$\cong T(r+\Delta r)\Delta\phi \tan\alpha$$

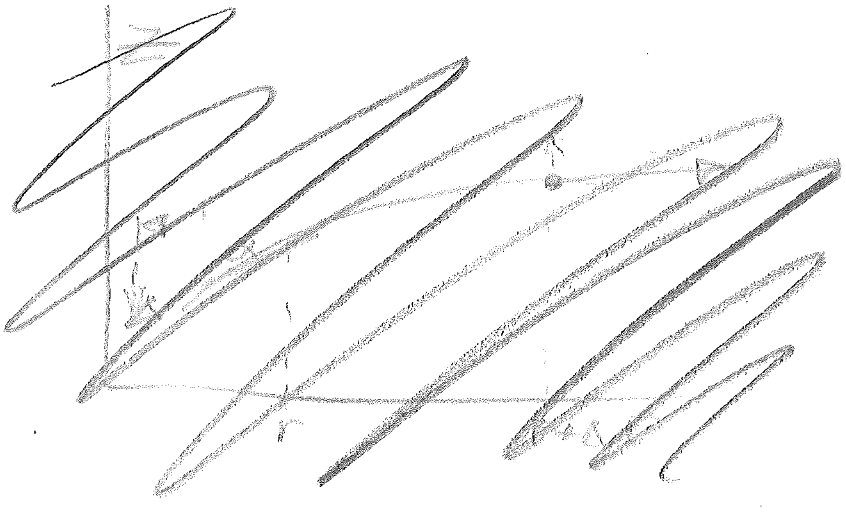
$$= T(r+\Delta r)\Delta\phi \left.\frac{\delta z}{\delta r}\right|_{r+\Delta r,\phi,t}$$

SIDE ②

THUS

$$(\Sigma F_z)_r = \underbrace{T(r+\Delta r)\Delta\phi \left.\frac{\delta z}{\delta r}\right|_{r+\Delta r,\phi,t}}_{\text{FOR SIDE 2}} - \underbrace{Tr\Delta\phi \left.\frac{\delta z}{\delta r}\right|_{r,\phi,t}}_{\text{FOR SIDE 1}}$$

20



4) ~~z(r~~

$$z(r,\phi,t) = a_m J_m(kr)\cos(m\phi+\phi_m)\cos(\omega t+\alpha)$$

NOTE

$$z(r,\phi,t) = z(r,\phi+p2\pi,t) \ni p=0,1,2,\ldots$$

$$\Rightarrow \cos(m\phi+\phi_m) = \cos\left[m(\phi+2\pi p)+\phi_m\right]$$

$$= \cos\left[(m\phi+\phi_m)+2\pi p\underline{m}\right]$$

THIS WILL HOLD FOR THE GENERAL CASE IF (pm) IS AN INTEGER. p HAS BEEN SPECIFIED AN INTEGER ⇒ m MUST BE AN INTEGER

✓ 10

$$5)\ Z_{mn} = A_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \cos(\omega_{mn} t + \alpha_{mn})$$

$m = 3, \quad n = 2$

~~Z_{mn}~~

$$Z_{32} = A_{32} \sin \frac{3\pi}{(0.1)} x \sin \frac{2\pi}{0.05} y \cos(\omega_{32} t + \alpha_{32})$$

$$\omega_{32} = \pi \sqrt{\frac{T}{\sigma}} \sqrt{\left(\frac{3}{0.1}\right)^2 + \left(\frac{2}{0.05}\right)^2} \quad \ni c = \sqrt{T/\sigma}$$

25

.05)100

$$= \pi \sqrt{\frac{T}{\sigma}} \sqrt{30^2 + \cancel{50^2}(40)^2}$$

$$2\cancel{\pi}(500) = \cancel{\pi} \sqrt{\frac{T}{(1)}} \sqrt{900 + 2500}$$

1600

$$\frac{1000}{\sqrt{\cancel{3400}\ 2500}} = \sqrt{T} = \frac{100}{10\sqrt{34}} = \frac{10}{\sqrt{34}}$$

$$T = \frac{100}{34} \frac{nt}{meter}$$

400

25

BOB MARKS

FINAL EXAM PROBLEMS   Nov. 21, 1972

1	1.00
2	0.95
3	0.00
4	1.00
5	1.00
6	1.00
7	0.60
8	0.80
9	0.15
10	1.00
11	0.90
12	0.60
	9.00

①. If an harmonic spherical wave of the form

$$\underset{\sim}{P} = \frac{A}{r} e^{i(\omega t - kr)}$$

exists in a fluid, then the volume occupied by a tiny fixed mass dm of fluid located at any point varies above and below some mean value V_0. Show that at any time t the difference ΔV between the instantaneous volume V of dm and its mean value V_0 is given by

$$\Delta V = -V_0 \frac{A}{\rho c^2} e^{i(\omega t - kr)}$$

② For a driven damped harmonic oscillator, vibrating in the steady state, is the rate at which the driving force supplies energy to the oscillator equal at every instant to the rate energy is being dissipated? Is the total mechanical energy (potential plus kinetic) of a driven damped harmonic oscillator a constant in the steady state?

3. A thin flexible wire is tied to two fixed supports a distance L apart and a sinusoidal current equal to the real part of $I_0 e^{i\omega t}$ is arranged to flow in the wire. A horse shoe magnet produces a uniform magnetic field of magnitude B over a short piece ΔL of the wire, the magnet being located a distance a from one end, as indicated in the figure. The force exerted on the segment ΔL due to the fact it is carrying current in a magnetic field is $BI_0 e^{i\omega t} \Delta L$. Assume this force sets up in the wire harmonic waves of the same frequency so that the motion of the wire lying between x=0 and x=a is described by some function $y_1(x,t)$ and that between x=a and x=L by some function $y_2(x,t)$.

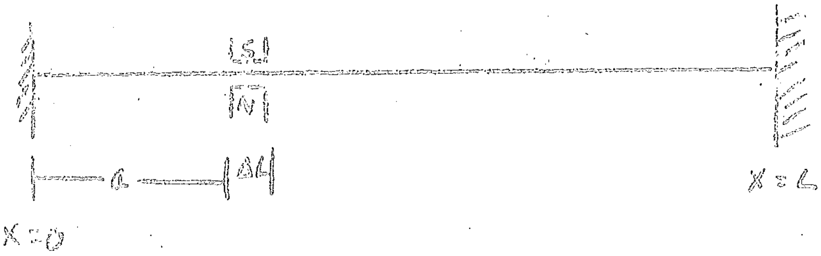
(i) By applying the boundary conditions show that $y_1(x,t)$ and $y_2(x,t)$ may be written in the form

$$y_1(x,t) = f_1(x)e^{i\omega t}$$

$$y_2(x,t) = f_2(x)e^{i\omega t}$$

(ii) Assuming the mass per unit length of the wire is σ, and the tension is T, write down the equation of motion for the segment ΔL of the wire. Now let $\Delta L \to 0$ and $B \to \infty$ so that the
(BAI_0) quantity [illegible] = F_0 remains finite. Use the resulting equation to determine any unknown constants in $f_1(x)$ and $f_2(x)$, i.e., find $y_1(x,t)$ and $y_2(x,t)$ in terms of F_0, $k = \omega/c$ a, L, T, x and t.

3. (iii) Imagine the magnet is moved to the center of wire, i.e. to $x = \frac{L}{2}$, and the frequency w of the current is varied. For what frequencies in terms of L_1 and c will the amplitude of the motion at the driving point $x = \frac{L}{2}$ be large?



④ Calculate the strength of a source consisting of a membrane of radius a and tension T mounted in an infinite baffle and vibrating in its fundamental mode with an amplitude A_0 at its center. Let σ be the mass per unit area of the membrane.

⑤ Are the characteristic frequencies of a membrane stretched over a square frame all integral multiples of the fundamental?

⑥ By assuming that each element dS of a piston vibrating in an infinite baffle contributes to the pressure at a point Q an amount

$$\underset{\sim}{dP} = \frac{i\rho c k U_0 \, dS}{2\pi r'} e^{i(\omega t - kr')} \qquad (i)$$

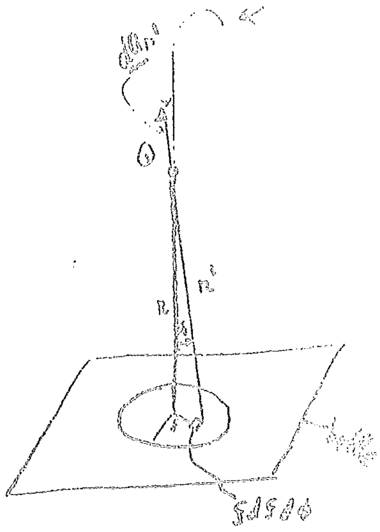
we were able to show that the pressure at a point Q on the axis of the piston was

$$\underset{\sim}{P} = -\rho c U_0 e^{i\omega t}\left[e^{-ik\sqrt{a^2+r^2}} - e^{-ikr}\right]$$

Since (i) represents a spherical wave (function of r' only) it would have associated with it a radial particle velocity $\underset{\sim}{du_{r'}}$ given by

$$\underset{\sim}{du_{r'}} = \frac{\underset{\sim}{dP}}{\underset{\sim}{z'}} \quad \text{where} \quad \underset{\sim}{z'} = \frac{\rho c k r'(kr' + i)}{1 + (kr')^2}$$

Unlike the pressure, the particle velocity at a point is a directed quantity, the particle velocity $du_{r'}$ due to dS being along the line joining dS and Q. At some instant it might be in the direction shown by the arrow labelled in the figure. It should be evident from symmetry, that at any instant of time the resultant particle velocity $\underline{u}$ at the point Q must be along the axis of the piston and hence would be given by



⑥ (continued) $\underset{\sim}{U} = \int\limits_{\substack{\text{Area} \\ \text{of Piston}}} d\underset{\sim}{U}_{r'} \cos\alpha = \int\limits_{\substack{\text{Area of} \\ \text{Piston}}} d\underset{\sim}{U}_{r'} \frac{r}{r'}$

(a) Using the fact that $r' = \sqrt{\xi^2 + r^2}$ and $dr' = \xi d\xi / \sqrt{\xi^2 + r^2}$

show that

$$\underset{\sim}{U} = r U_0 e^{i\omega t} \int_{r}^{\sqrt{r^2+a^2}} \left\{ ik \frac{e^{-ikr'}}{r'} + \frac{e^{-ikr'}}{r'^2} \right\} dr'$$

which may be integrated by the use of tables (e.g. Pierce, <u>A Short Table of Integrals</u>, Formula 404) yielding

$$\underset{\sim}{U} = U_0 e^{i\omega t} \left[e^{-ikr} - \frac{r e^{-ik\sqrt{r^2+a^2}}}{\sqrt{r^2+a^2}} \right]$$

(b) Find the real part of $\underset{\sim}{u}$ at point Q.

(c) Using the real part of $\underset{\sim}{P}$ at point Q derived in class find the intensity at Q by evaluating

$$I_Q = \frac{1}{T} \int_0^T P_{real} \, U_{real} \, dt$$

It is readily shown that

$$\frac{1}{T} \int_0^T \sin(\omega t - \alpha) \cos(\omega t - \beta) \, dt = \frac{1}{2} \sin(\beta - \alpha)$$

which might prove helpful.

⑦ A thin metal steel disk of 1 cm thickness is supported by a light spring fastened to the top of a pipe partially filled with water. The disk is positioned vertically so that it is located at the air water interface. If a <u>plane</u> harmonic wave $\underset{\sim}{P}_i$ is incident on the disk from the water side, the disk will be set in motion and in the steady state waves will exist in both the lower and upper portions of the pipe. If the top of the pipe is fitted with an absorber that absorbs any wave energy incident on it only a single wave $\underset{\sim}{P}_t$ will exist in the region above the disk. The harmonic waves $\underset{\sim}{P}_i$, $\underset{\sim}{P}_r$ and $\underset{\sim}{P}_t$ give rise to harmonic forces on the disk and the disk spring system can be thought of as an harmonic oscillator having a mechanical impedance,

$$\underset{\sim}{Z}_m = R_m + i(\omega m - K/\omega)$$

(7.) (continued)

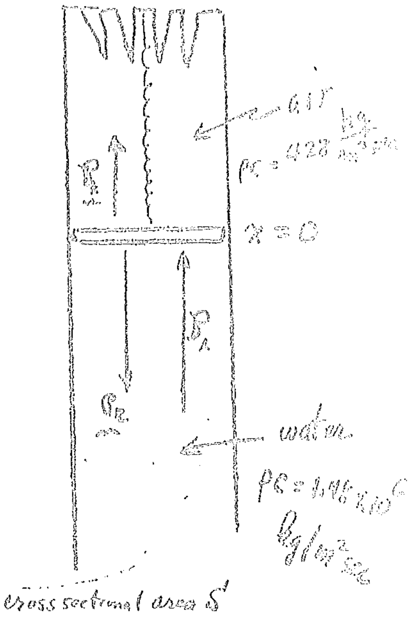
Suppose the frequency of the waves is 5000 hertz and suppose for this frequency

$$\underset{\sim}{Z}_m \doteq i\omega m$$

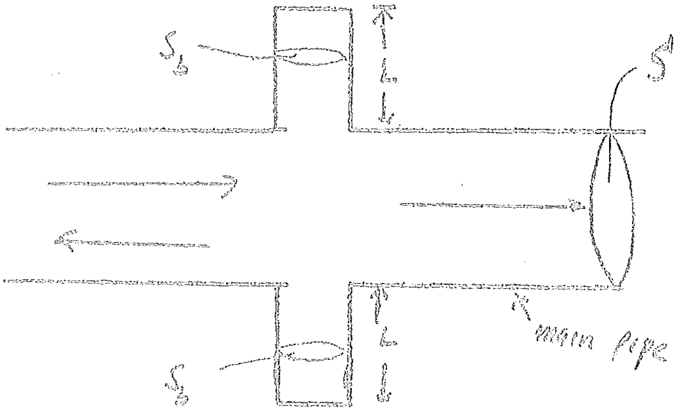
(1) Write down suitable expressions for the harmonic waves $\underset{\sim}{P}_i$, $\underset{\sim}{P}_r$, $\underset{\sim}{P}_t$

(2) Write down in terms of $\underset{\sim}{P}_i$, $\underset{\sim}{P}_r$, $\underset{\sim}{P}_t$, S, w and m, the velocity $\dot{\underset{\sim}{x}}_p$ of the disk.

(3) Use the fact that the particle velocity evaluated at x=0 and the velocity of the disk must be the same to determine the ratio of the amplitudes of $\underset{\sim}{P}_r$ and $\underset{\sim}{P}_i$, and the phase shift that occurs on reflection.



(8.) Calculate the sound power transmission coefficient for the case illustrated.



9. Kinsler & Frey 8.2.

(10.) Kinsler & Frey 8.3.

(11.) Kinsler & Frey 8.13.

(12.) Problem 7 on hand out sheet entitled Problems for Chap VI.

1) LET V_0 AND ρ_0 BE THE EQUILIBRIUM PARAMETERS FOR dm, AND V_1 AND ρ_1 DESCRIBE dm AT SOME TIME t. THEN:

$$dm = \rho_0 V_0 = \rho_1 V_1$$
$$\frac{\rho_1}{\rho_0} = \frac{V_0}{V_1}$$
$$\frac{\rho_1}{\rho_0} - 1 = \frac{V_0}{V_1} - 1$$
$$\frac{\rho_1 - \rho_0}{\rho_0} = \frac{V_0 - V_1}{V_1}$$

$\frac{\rho_1 - \rho_0}{\rho} \triangleq s$, THE CONDENSATION (pg 109), WHICH IS, FOR A SPHERICAL WAVE IS PROPORTIONAL TO THE PRESSURE:

$$s = \frac{p}{\rho_0 c^2} \qquad \text{(pg 158)}$$

THUS

$$s = \frac{p}{\rho_0 c^2} = \frac{\rho_1 - \rho_0}{\rho_0} = \frac{V_0 - V_1}{V_1}$$

$$\Rightarrow \frac{V_0 - V_1}{V_1} = \frac{p}{\rho_0 c^2}$$

$V_1 - V_0 = \Delta V << 1 \Rightarrow V_1 \sim V_0$

THEN $\frac{-\Delta V}{V_0} = \frac{p}{\rho_0 c^2}$

$$\Delta V = \frac{-V_0 p}{\rho_0 c^2} \quad ; \quad p = \frac{A}{r} e^{i(\omega t - kr)}$$

$$= -V_0 \frac{A}{r \rho_0 c^2} e^{i(\omega t - kr)}$$ ✓

[NOTE: SOLUTION GIVEN IN PROBLEM IS NOT DIMENSIONALLY CONSISTANT. INTUITIVELY, IT SEEMS THE FARTHER ONE MOVES FROM THE SOURCE, THE SMALLER ΔV BECOMES, ie ΔV IS A MONOTONICALLY DECREASING FUNCTION OF r] This seem's reasonable to me. If you are sufficiently far from the source, then the volume of any fixed mass of fluid would remain essentially unchanged just as it would if no wave were present.

2)a) FOR A DRIVEN DAMPED HARMONIC OSCILLATOR IN THE STEADY STATE:

$$V = \frac{1}{Z_m} F_0 \cos(\omega t - \phi)$$

WHERE $Z_m = \sqrt{R_m^2 + (\omega m - k/\omega)^2}$

$$\phi = \tan^{-1} \frac{\omega m - k/\omega}{R_m}$$

ALSO: $F = F_0 \cos \omega t$

THE RATE AT WHICH THE DRIVING FORCE IS SUPPLYING ENERGY IS EQUAL TO THE PRODUCT OF THE INSTANTANEOUS DRIVING FORCE AND THE RESULTING VELOCITY:

$$W_{i_{IN}} = F V = \frac{F_0^2}{Z_m} \cos \omega t \cos(\omega t - \phi)$$

THE RATE AT WHICH ENERGY IS ABSORBED IN THE SYSTEM IS THE RATE ENERGY IS DISSIPATED BY R_m:

$$W_{i_{OUT}} = \frac{1}{2} R_m (\dot{x})^2 = \frac{1}{2} R_m V^2$$

$$= \frac{R_m F_0^2}{Z_m^2} \cos^2(\omega t - \phi)$$

THE TOTAL WORK DONE PER VIBRATION BY THE DRIVING FORCE:

$$W_{IN} = \frac{\int_0^\tau W_{i_{IN}} dt}{\tau} \quad \ni \quad \tau = \frac{2\pi}{\omega}$$

$$= \frac{F^2}{Z_m \tau} \int_0^\tau \cos \omega t \cos(\omega t - \phi)\, dt$$

$$= \frac{F^2}{Z_m \tau} \int_0^\tau [\cos^2 \omega t \cos \phi + \cos \omega t \sin \omega t \sin \phi]\, dt$$

$$= \frac{F^2}{2 Z_m} \cos \phi$$

SINCE $\cos\phi = \frac{R_m}{Z_m}$ (MECHANICAL PWR. FACTOR)

$$W_{IN} = \frac{F^2 R_m}{2 Z_m^2}$$

THE DISSIPATED POWER AVERAGED OVER A CYCLE IS:

$$W_{OUT} = \frac{\int_0^{\tau} W'_{OUT}\,dt}{\tau}$$

$$= \frac{F^2 R_m}{\tau Z_m^2} \int_0^{\tau} \cos^2(\omega t - \phi)\,dt$$

$$= \frac{F^2 R_m}{2 Z_m^2}$$

THUS, IN THE STEADY STATE, THE AMPLITUDE AND PHASE OF A DRIVEN OSCILLATOR SO ADJUST THEMSELVES THAT THE AVERAGE POWER BEING SUPPLIED BY THE DRIVING FORCE IS JUST EQUAL TO THAT BEING DISSIPATED BY THE FRICTIONAL FORCE

THE MOTION OF A DRIVEN DAMPED HARMONIC OSCILLATOR IS GIVEN BY:

$$X = -\frac{i(F_0/\omega)\,e^{i\omega t}}{R + i(\omega m - K/\omega)} \Rightarrow Re(x) = \frac{F_0/\omega \sin(\omega t - \phi)}{\sqrt{R^2 + (\omega m - K/\omega)}}$$

$$\frac{dx}{dt} = V = \frac{1}{Z_m} F_0 e^{i\omega t} \Rightarrow Re(\dot{x}) = F_0 \frac{\cos(\omega t - \phi)}{\sqrt{R^2 + (\omega m - K/\omega)}}$$

THE KINETIC ENERGY IS THEN $(\frac{1}{2} m V^2)$

$$U_K = \frac{1}{2} m \dot{x}_{REAL}^2$$

$$= \frac{m F_0^2}{2 Z_m^2} \cos^2(\omega t - \phi)$$

THE POTENTIAL ENERGY $(= \frac{1}{2} k x^2)$

$$U_P = \frac{1}{2} k X^2$$

$$= \frac{K F_0^2}{2 Z_m^2 \omega^2} \sin^2(\omega t - \phi)$$

THEN:

$$U_K + U_P = \frac{F_0^2}{2 Z_m^2}\left[m \cos^2(\omega t - \phi) + \frac{K}{\omega^2} \sin^2(\omega t - \phi)\right]$$

WHICH IS CONSTANT @ RESONANCE, ✓

OR @ $m = K/\omega_R^2 \Rightarrow \omega_R = \sqrt{K/m}$

AND $Z_m = R$

AT RESONANCE:

$$U_K + U_P = \frac{F_0^2}{2 R_m^2}$$ ✓

4) THE MOTION OF A CIRCULAR MEMBRANE VIBRATING IN IT'S FUNDEMENTAL MODE IS;

$z_0 = A_0 \, J_0\left(\frac{z_1}{a} r\right) \cos\left(\frac{z_1}{a} ct + \Omega_{01}\right) \ni z_{01} = 2.405$

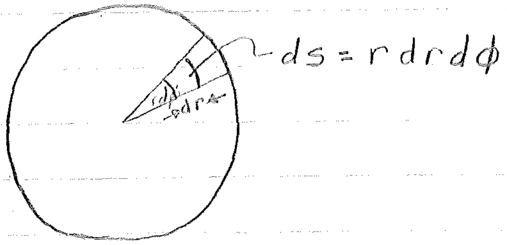
THEN: $\frac{dz_0}{dt} = \frac{-A_0 z_1 c}{a} J_0\left(\frac{z_1}{a} r\right) \sin\left(\frac{z_1}{a} ct + \Omega_{01}\right)$

THE VELOCITY AMPLITUDE IS THEN:

$U = + \frac{A_0 z c}{a} J_0\left(\frac{z_1}{a} r\right)$

THE SOURCE STRENGTH IS DEFINED AS:

$$Q = \int_S U \cdot ds$$



THE MEMBRANE'S MOTION IS NORMAL TO THE PLANE IT OCCUPIES $\Rightarrow U \cdot ds = U ds$

$$\therefore Q = \int_S + \frac{A_0 z_1 c}{a} J_0\left(\frac{z_1}{a} r\right) r\,dr\,d\phi$$

$$= + A_0 c \int_0^{2\pi} d\phi \int_0^a \frac{z_1 r}{a} J_0\left(\frac{z_1}{a} r\right) dr$$

LET $\gamma = \frac{z_1 r}{a} \Rightarrow d\gamma = \frac{z_1}{a} dr$

$$\Rightarrow Q = + A_0 c \, (2\pi) \int_0^{z_1} \gamma \, J_0(\gamma) \left(\frac{a}{z_1}\right) d\gamma$$

$$= \frac{+A_0 a c 2\pi}{z_1} \int_0^{z_1} \gamma \, J_0(\gamma) \, d\gamma$$

$$= \frac{+A_0 a c 2\pi}{z_1} \Big[\gamma \, J_1(\gamma) \Big]_0^{z_1}$$

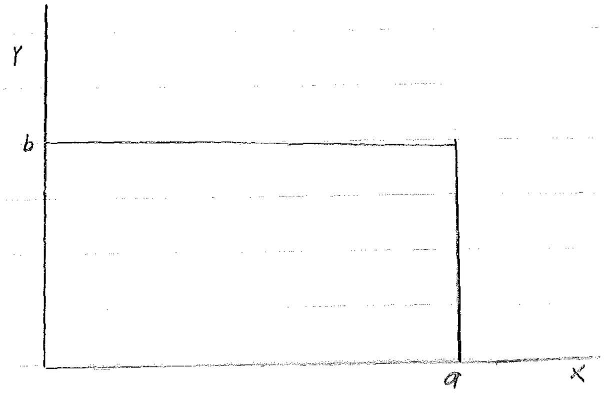
$$= \frac{+A_0 2\pi a c}{z_1} \Big[z_1 J_1(z_1) \Big]$$

$$= + A_0 2\pi a c J_1(z_1)$$

$c = \sqrt{T/\sigma}$; $J_1(z_1) = J_1(2.405) \simeq \frac{1}{2}$ ✓

$$\Rightarrow Q = A_0 \pi a \sqrt{T/\sigma}$$ ✓

5)



FOR A RECTANGULAR BOUNDRY, THE EIGEN FREQUENCIES ARE:

$$f_{mn} = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

FOR A SQUARE BOUNDRY, $a=b$ AND

$$f_{mn} = \frac{c}{2a}\sqrt{m^2+n^2}.$$

THE FUNDEMENTAL FREQUENCY $(m=n=1)$

$$f_{11} = \frac{c\sqrt{2}}{2a}$$

$$\frac{f_{mn}}{f_{11}} = \frac{\frac{c}{2a}\sqrt{m^2+n^2}}{\frac{c\sqrt{2}}{2a}} = \frac{\sqrt{m^2+n^2}}{\sqrt{2}}$$

THUS, THE CHARACTERISTIC FREQUENCIES OF A STRETCHED MEMBRANE WITH A SQUARE FRAME ARE NOT IN GENERAL INTEGRAL MULTIPLES OF THE FUNDEMENTAL ✓

6)a) $U = \int dU_{r'} \frac{r}{r'}$

$$dU_{r'} = \frac{dP}{Z'} = \left[\frac{i\,\cancel{\rho c k}\,U_0\,ds}{2\pi r'}e^{i(\omega t-kr')}\right]\left[\frac{1+(kr')^2}{\cancel{\rho c k}\,r'(kr'+i)}\right]$$

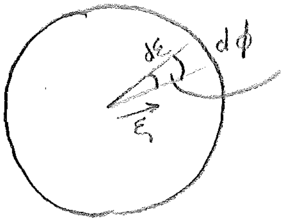
$$= \frac{i\,U_0\,ds\,[1+(kr')^2]}{2\pi r'^2\,[kr'+i]}e^{i(\omega t-kr')}$$

$$= \frac{i\,U_0\,ds\,\cancel{(kr'+i)}(kr'-i)}{2\pi r'^2\,\cancel{(i+kr')}}e^{i(\omega t-kr')}$$

$$= \frac{i\,U_0\,ds\,(kr'-i)}{2\pi r'^2}e^{i(\omega t-kr')}$$

$$= \frac{U_0(1+ikr')\,ds}{2\pi r'^2}e^{i(\omega t-kr')}$$

$$U = \int_{PISTON}\frac{r\,U_0(1+ikr')\,ds}{2\pi r'^3}e^{i(\omega t-kr')}$$



$ds = \xi\,d\phi\,d\xi$

$$U = r\,U_0\,e^{i\omega t}\int_0^a\int_0^{2\pi}\frac{(1+ikr')}{2\pi r'^3}e^{-ikr'}\,\xi\,d\phi\,d\xi$$

NOW $\quad r' = \sqrt{\xi^2+r^2}$

$$dr' = \frac{\xi\,d\xi}{\sqrt{\xi^2+r^2}} = \frac{ds}{r'\,d\phi}$$

$$\Rightarrow ds = r'\,dr'\,d\phi \qquad \therefore\ \xi=0 \Rightarrow r'=r;\ \xi=a \Rightarrow r'=\sqrt{a^2+r^2}$$

$$U = r\,U_0\,e^{i\omega t}\int_0^{2\pi}\int_r^{\sqrt{a^2+r^2}}\frac{1+ikr'}{2\pi r'^3}e^{-ikr'}\,r'\,dr'\,d\phi$$

$$= r\,U_0\,e^{i\omega t}\int_r^{\sqrt{a^2+r^2}}\frac{1+ikr'}{r'^2}e^{-ikr'}\,dr'$$

$$= r\,U_0\,e^{i\omega t}\int_r^{\sqrt{a^2+r^2}}\left[ik\frac{e^{-ikr'}}{r'}+\frac{e^{ikr'}}{r'^2}\right]dr'$$

$$
\begin{aligned}
\text{b)}\quad U &= U_0 e^{i\omega t}\left[e^{-ikr} - \frac{r e^{-ik\sqrt{r^2+a^2}}}{\sqrt{r^2+a^2}}\right] \\
&= U_0\left[e^{i(\omega t-kr)} - \frac{r}{\sqrt{r^2+a^2}}\, e^{i[\omega t-k\sqrt{r^2+a^2}]}\right] \\
&= U_0\Big[\cos(\omega t-kr) - \frac{r}{\sqrt{r^2+a^2}}\cos\{\omega t-k\sqrt{r^2+a^2}\} \\
&\qquad +j\Big\{\sin(\omega t-kr) - \frac{r}{\sqrt{r^2+a^2}}\sin\{\omega t-k\sqrt{r^2+a^2}\}\Big\}\Big]
\end{aligned}
$$

$$
\Rightarrow R_e[U] = U_0\left[\cos(\omega t-kr) - \frac{r}{\sqrt{r^2+a^2}}\cos(\omega t-k\sqrt{r^2+a^2})\right]
$$

$$
\begin{aligned}
\text{c)}\quad P &= -\rho c U_0 e^{i\omega t}\left[e^{-ik\sqrt{a^2+r^2}} - e^{-ikr}\right] \\
&= -\rho c U_0\left[e^{i(\omega t-k\sqrt{a^2+r^2})} - e^{i(\omega t-kr)}\right] \\
&= \rho c U_0\Big[\cos(\omega t-kr) - \cos(\omega t-k\sqrt{r^2+a^2}) \\
&\qquad +j\ \sin(\omega t-kr) - j\sin(\omega t-k\sqrt{r^2+a^2})\Big]
\end{aligned}
$$

$$
\Rightarrow R_e[P] = \rho c U_0\left[\cos(\omega t-kr) - \cos(\omega t-k\sqrt{r^2+a^2})\right]
$$

$$I_a = \frac{1}{\tau}\int_0^{\tau} P_{REAL}\ U_{REAL}\, dt$$

$$= \frac{1}{\tau}\int_0^{\tau}\Big[\rho c U_0\{\cos(\omega t - kr) - \cos(\omega t - k\sqrt{r^2+a^2})\}\Big]$$

$$\Big[U_0\{\cos(\omega t - kr) - \frac{r}{\sqrt{r^2+a^2}}\cos(\omega t - k\sqrt{r^2+a^2}\}\Big]$$

$$dt$$

$$= \frac{\rho c U_0^2}{\tau}\int_0^{\tau}\Big[\cos^2(\omega t - kr)$$

$$+ \frac{r}{\sqrt{r^2+a^2}}\cos^2(\omega t - k\sqrt{r^2+a^2})$$

$$- \cos(\omega t - kr)\frac{r}{\sqrt{r^2+a^2}}\cos(\omega t - k\sqrt{r^2+a^2})$$

$$- \cos(\omega t - kr)\cos(\omega t - k\sqrt{r^2+a^2})\Big]dt$$

$$= \frac{\rho c U_0^2}{\tau}\int_0^{\tau}\Big[\frac{1}{2}\{1 + \cos 2(\omega t - kr)\}$$

$$+ \frac{r}{2\sqrt{r^2+a^2}}\{1 + \cos 2(\omega t - k\sqrt{r^2+a^2}\}$$

$$- \Big[\frac{r}{\sqrt{r^2+a^2}} + 1\Big]\cos(\omega t - kr)\cos(\omega t - k\sqrt{r^2+a^2}\}\Big]dt$$

$$= \frac{\rho c U_0^2}{\tau}\Big[\int_0^{\tau}\frac{1}{2}\Big(1 + \frac{r}{\sqrt{r^2+a^2}}\Big)dt$$

$$- \Big(\frac{r}{\sqrt{r^2+a^2}} + 1\Big)\int_0^{\tau}\cos(\omega t - kr)\cos(\omega t - k\sqrt{r^2+a^2})dt\Big]$$

$$= \frac{\rho c U_0^2}{\tau}\Big[\frac{\tau}{2}\Big(1 + \frac{r}{\sqrt{r^2+a^2}}\Big)\Big]$$

$$- \rho c U_0^2\Big(\frac{r}{\sqrt{r^2+a^2}} + 1\Big)\Big(\frac{1}{\tau}\int_0^{\tau}\sin(\omega t \overset{-(kr-\frac{\pi}{2})}{- kr + \frac{\pi}{2}})$$

$$\cos(\omega t - k\sqrt{r^2+a^2})dt\Big)$$

$$= \frac{\rho c U_0^2}{2}\Big(1 + \frac{r}{\sqrt{r^2+a^2}}\Big)$$

$$- \rho c U_0^2\Big(1 + \frac{r}{\sqrt{r^2+a^2}}\Big)\Big(\frac{1}{2}\sin\{k\sqrt{r^2+a^2} - kr + \frac{\pi}{2}\}\Big)$$

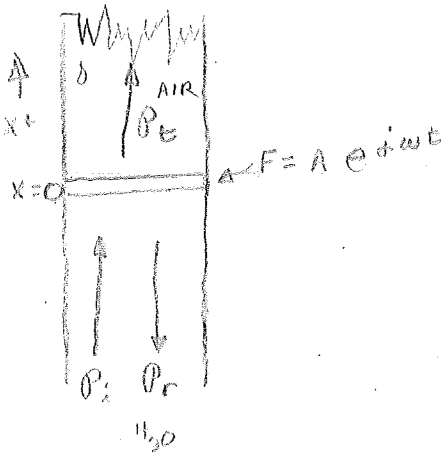
$$= \frac{1}{2}\rho c U_0^2\Big(1 + \frac{r}{\sqrt{r^2+a^2}}\Big)$$

$$- \frac{1}{2}\rho c U_0^2\Big(1 + \frac{r}{\sqrt{r^2+a^2}}\Big)\cos\Big[k(\sqrt{r^2+a^2} - r)\Big]$$

$$= \frac{1}{2}\rho c U_0^2\Big(1 + \frac{r}{\sqrt{r^2+a^2}}\Big)\Big[1 - \cos\{k(\sqrt{r^2+a^2} - r)\}\Big]$$

✓

7) a)



$; \; Z_m = i\omega m$

$$P_i = A_1 e^{i(\omega t - k_1 x)}$$
$$P_r = B_1 e^{i(\omega t + k_1 x)}$$
$$P_t = A_2 e^{i(\omega t - k_2 x)}$$

b) $\Sigma P|_{x=0} = P_i|_{x=0} + P_t|_{x=0} + P_r|_{x=0}$

SO THE TOTAL FORCE ON THE DISC IS:

$$F_D = S\,[P_i|_{x=0} + P_t|_{x=0} + P_r|_{x=0}]$$
$$a_D = \frac{F_D}{m} = \frac{S}{m}[P_i|_{x=0} + P_t|_{x=0} + P_r|_{x=0}]$$
$$V_D = \int a_D\,dt = \int \frac{S}{m}[P_i|_{x=0} + P_t|_{x=0} + P_r|_{x=0}]\,dt$$
$$= \int \frac{S}{m}[A_1 + B_1 + A_2]\,e^{i\omega t}\,dt$$

$$\dot{x}_D = \frac{-iS}{\omega m}[P_i \overset{?}{\boxed{+}} P_t + P_r]_{x=0}$$

$$\dot{x} = \frac{[(P_i + P_r) - P_t]\,S}{i\omega m}$$

when $(P_i + P_r) > 0$ force is in $+x$ direction

When P_t is > 0 force is in the $-x$ direction

$$\frac{A_1}{\rho_1 c_1} - \frac{B_1}{\rho_1 c_1} = \frac{A_1 + B_1 - A_2}{i\omega m} = \frac{A_2}{\rho_2 c_2}$$

Solve for

$$\frac{B_1}{A_1} = \frac{(\rho_2 c_2 - \rho_1 c_1)S + i\omega m}{(\rho_2 c_2 + \rho_1 c_1)S + i\omega m} \Rightarrow \left|\frac{B_1}{A_1}\right| = 1$$

$$\frac{B_1}{A_1} = \left|\frac{B_1}{A_1}\right| e^{i\phi}$$

$\phi \simeq$ [illegible]°

c) $P_{UP} = P_t = A_2 e^{i(\omega t - k_2 x)}$

$$\Rightarrow U_{UP} = \frac{A_2}{\rho_2 c_2} e^{i(\omega t - k_2 x)} = \frac{1}{\rho_2 c_2} P_t \quad ✓$$

$$P_{DOWN} = A_1 e^{i(\omega t - k_1 x)} + B_1 e^{i(\omega t - k_1 x)}$$

$$U_{DOWN} = \frac{1}{\rho_1 c_1}\left[A_1 e^{i(\omega t - k_1 x)} - B_1 e^{i(\omega t - k_1 x)}\right]$$

$$\underbrace{\qquad\qquad}= \frac{1}{\rho_1 c_1}\left[P_i - P_r\right] \quad ✓$$

FIND B_1/A_1

$$U_{DOWN}\Big|_{x=0} = \dot{X}_D$$

$$\frac{1}{\rho_1 c_1}\left[A_1 - B_1\right] \cancel{e^{i\omega t}} = \frac{-is}{\omega m}\left[A_1 + B_1 + A_2\right]\cancel{e^{i\omega t}}$$

$$\frac{i\omega m}{s\rho_1 c_1}\left[A_1 - B_1\right] = A_1 + B_1 \oplus A_2 \quad \longleftarrow Ⓐ$$

$$U_{UP}\Big|_{x=0} = \dot{X}_D$$

$$\frac{A_2}{\rho_2 c_2} e^{i\omega t} = \frac{s}{i\omega m}\left[A_1 + B_1 + A_2\right] e^{i\omega t}$$

$$\frac{i\omega m A_2}{s\rho_2 c_2} - A_2 = A_1 + B_1$$

$$A_2\left(-1 + \frac{i\omega m}{s\rho_2 c_2}\right) = A_1 + B_1$$

$$A_2 = \frac{A_1 + B_1}{-1 + \frac{i\omega m}{s\rho_2 c_2}}$$

PLUGGING INTO Ⓐ

$$\frac{i\omega m}{s\rho_1 c_1}\left[A_1 - B_1\right] = A_1 + B_1 + \frac{A_1 + B_1}{-1 + \frac{i\omega m}{s\rho_2 c_2}}$$

$$\frac{i\omega m}{s\rho_1 c_1}\left[A_1 - B_1\right] = \left[1 - \frac{1}{1 - \frac{i\omega m}{s\rho_2 c_2}}\right]\left[A_1 + B_1\right]$$

$$\left[\frac{i\omega m}{s\rho_1 c_1} - 1 + \frac{1}{1 - \frac{i\omega m}{s\rho_2 c_2}}\right] A_1 = \left[1 - \frac{1}{\left(1 - \frac{i\omega m}{s\rho_2 c_2}\right)} + \frac{i\omega m}{s\rho_1 c_1}\right] 1$$

$$\frac{B_1}{A_1} = \frac{\frac{i\omega m}{s\rho_1 c_1} - \left[1 - \frac{1}{\left(1 - \frac{i\omega m}{s\rho_2 c_2}\right.}\right]}{\frac{i\omega m}{s\rho_1 c_1} + \left[1 - \frac{1}{\left(1 - \frac{i\omega m}{s\rho_2 c_2}\right.}\right]}$$

$$\frac{B_1}{A_1} = \frac{\frac{i\omega m}{s\rho_1 c_1} - \left[1 - \frac{s\rho_2 c_2}{s\rho_2 c_2 - i\omega m}\right]}{\frac{i\omega m}{s\rho_1 c_1} + \left[1 - \frac{s\rho_2 c_2}{s\rho_2 c_2 - i\omega m}\right]}$$

$$= \frac{\frac{i\omega m}{s\rho_1 c_1} - \left[\frac{s\rho_2 c_2 - i\omega m - s\rho_2 c_2}{s\rho_2 c_2 - i\omega m}\right]}{\frac{i\omega m}{s\rho_1 c_1} + \left[\frac{s\rho_2 c_2 - i\omega m - s\rho_2 c_2}{s\rho_2 c_2 - i\omega m}\right]}$$

$$= \frac{\frac{i\omega m}{s\rho_1 c_1} + \frac{i\omega m}{s\rho_2 c_2 - i\omega m}}{\frac{i\omega m}{s\rho_1 c_1} - \frac{i\omega m}{s\rho_2 c_2 - i\omega m}}$$

$$= \frac{\frac{1}{s\rho_1 c_1} + \frac{1}{s\rho_2 c_2 - i\omega m}}{\frac{1}{s\rho_1 c_1} - \frac{1}{s\rho_2 c_2 - i\omega m}}$$

$$= \frac{\frac{s\rho_2 c_2 - i\omega m}{s\rho_1 c_1} + 1}{\frac{s\rho_2 c_2 - i\omega m}{s\rho_1 c_1} - 1}$$

$$= \frac{s\rho_2 c_2 - i\omega m + s\rho_1 c_1}{s\rho_2 c_2 - i\omega m - s\rho_1 c_1}$$

$$= \left[\frac{s(\rho_2 c_2 + \rho_1 c_1) - i\omega m}{s(\rho_2 c_2 - \rho_1 c_1) - i\omega m}\right]\left[\frac{s(\rho_2 c_2 - \rho_1 c_1) + i\omega m}{s(\rho_2 c_2 - \rho_1 c_1) + i\omega m}\right]$$

$$= \frac{s^2(\rho_2 c_2 + \rho_1 c_1)(\rho_2 c_2 - \rho_1 c_1) + i\omega m s(\rho_2 c_2 + \rho_1 c_1 - \rho_2 c_2 + \rho_1 c_1) + \omega^2 m^2}{[s(\rho_2 c_2 - \rho_1 c_1)]^2 + \omega^2 m^2}$$

$$= \frac{s^2[(\rho_2 c_2)^2 - (\rho_1 c_1)^2] + (\omega m)^2 + i2\omega m s\rho_1 c_1}{s^2(\rho_2 c_2 - \rho_1 c_1)^2 + \omega^2 m^2}$$

SO THERE WILL BE A PHASE SHIFT OF:

$$\operatorname{atan} \frac{2 \omega m S \rho_1 c_1}{S^2 [(\rho_2 c_2)^2 - (\rho_1 c_1)^2]} ?$$

$$= \operatorname{atan} \frac{2 \omega m}{S [(\rho_2 c_2)^2 - (\rho_1 c_1)^2]}$$

$m = \rho_{stee} t S$ where t is the thickness

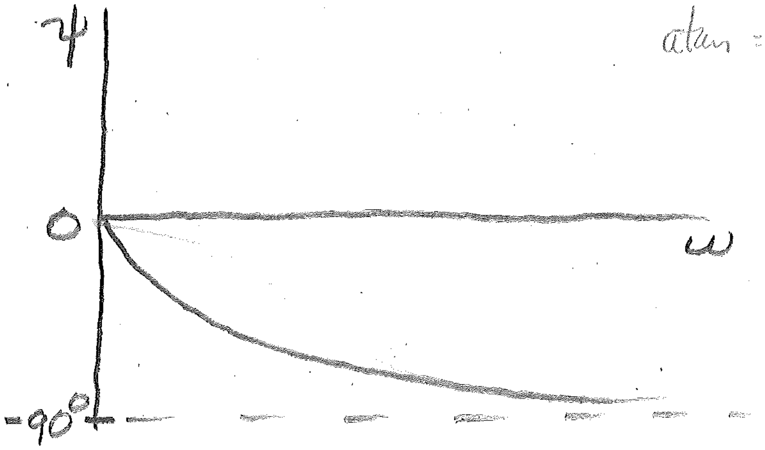
FOR A DISK WITH A SMALL MASS ?

PHASE SHIFT = $\psi \sim 0^0$

AS THE MASS INCREASES, ψ DECREASES [SINCE $\rho_1 c_1 > \rho_2 c_2$] TO MAXIMUM (MINIMUM?) VALUE OF $-90°$.

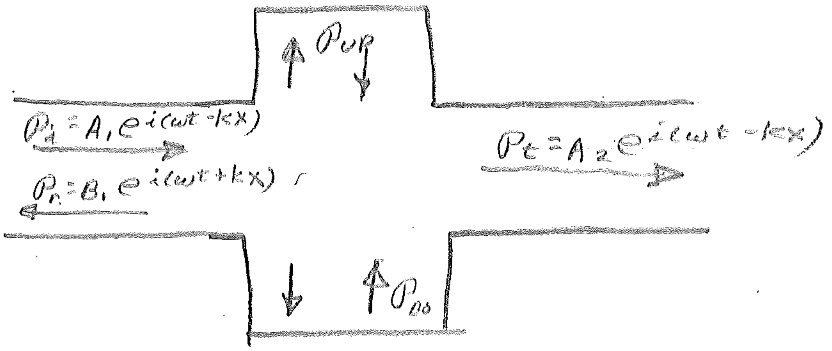
THE PHASE SHIFT ALSO DECREASES WITH ω

$$\operatorname{atan} = \frac{2 \omega \rho_s \rho_1 c_1 t}{[(\rho_2 c_2)^2 - (\rho_1 c_1)^2] + (\omega m)^2}$$



For very large Mass $\underset{\sim}{u_i} + \underset{\sim}{u_r} = 0$ at $x=0$ which leads to $\vec{A}_1 = \vec{B}_1$ & hence no phase shift in pressure wave.

8)



$$P_L|_{x=0} = P_{UP}|_{y=0} = P_D|_{y=0} = P_t|_{x=0}$$

$$U_L|_{x=0} S = U_{UP}|_{y=0} S_b + U_{DOWN}|_{y=0} S_b + U_R|_{x=0} S$$

$$\left[\frac{U_L S}{P_L} = \frac{U_R S}{P_r} + \frac{U_{UP} S_b}{P_U} + \frac{U_D S_b}{P_D}\right]_{x=y=0}$$

$$\frac{1}{P_L/U_L S} = \frac{1}{P_L/U_R S} + \frac{1}{P_U/U_{UP} S_b} + \frac{1}{P_D/U_D S_b}$$

$$\frac{1}{Z_L} = \frac{1}{Z_R} + \frac{1}{Z_{UP}} + \frac{1}{Z_{DOWN}} = \frac{1}{Z_R} + \frac{1}{\left(\frac{Z_u}{2}\right)}$$

$$Z_L = \frac{Z_{UP} Z_{DOWN} + Z_R Z_{DOWN} + Z_R Z_{UP}}{Z_r Z_{UP} Z_{DOWN}}$$

$$\frac{\rho c}{S} \frac{A_1 + B_1}{A_1 - B_1} = \frac{Z_{UP} Z_D + \frac{\rho c}{S} Z_D + \frac{\rho c}{S} Z_{UP}}{Z_U Z_D \rho c/S}$$

FROM SYMMETRY: $Z_U = Z_D$

$$\frac{\rho c}{S} \frac{A_1 + B_1}{A_1 - B_1} = \frac{Z_D^2 + \frac{2\rho c}{S} Z_D}{\frac{\rho c}{S} Z_D^2}$$

$$\frac{A_1 + B_1}{A_1 - B_1} = \frac{\frac{S}{\rho c} Z_D^2 + 2 Z_D}{\frac{\rho c}{S} Z_D^2} = \frac{\frac{S}{\rho c} Z_D + 2}{\frac{\rho c}{S} Z_D}$$

If you call $\frac{Z_u}{2} = Z_e$ then $\frac{1}{Z_L} = \frac{1}{Z_R} + \frac{1}{Z_{eff}}$

$Z_L = \frac{Z_R Z_{eff}}{Z_R + Z_{eff}}$ and prob is identical with single branch with cap!

$$\frac{\rho c}{S} Z_0 (A_1 + B_1) = \left(\frac{S}{\rho c} Z_0 + 2\right)(A_1 - B_1)$$

$$B_1\left[\frac{\rho c}{S} Z_0 + \frac{S}{\rho c} Z_0 + 2\right] = A_1\left[\frac{S}{\rho c} Z_0 + 2 - \frac{\rho c}{S} Z_0\right]$$

$$\frac{B_1}{A_1} = \frac{\left(\frac{S}{\rho c} - \frac{\rho c}{S}\right) Z_0 + 2}{\left(\frac{S}{\rho c} + \frac{\rho c}{S}\right) Z_0 + 2}$$

$$A_1 + B_1 = A_2 \Rightarrow 1 + \frac{B_1}{A_1} = \frac{A_2}{A_1}$$

$$\Rightarrow \frac{A_2}{A_1} = \frac{\left(\frac{S}{\rho c} - \frac{\rho c}{S}\right) Z_0 + 2 + \left(\frac{S}{\rho c} + \frac{\rho c}{S}\right) Z_0 + 2}{\left(\frac{S}{\rho c} + \frac{\rho c}{S}\right) Z_0 + 2}$$

$$= \frac{\frac{2S}{\rho c} Z_0 + 4}{\left(\frac{S}{\rho c} + \frac{\rho c}{S}\right) Z_0 + 2}$$

FOR CAPPED TOP: $Z_b\big|_{y=0} = -\frac{i\rho c}{S_b} \cot kL$

✓

$$\cancel{\Rightarrow \frac{A_2}{A_1} = \frac{\frac{2S}{\rho c}\left[-\frac{i\rho c}{S_b}\right] + 4}{\left(\frac{S}{\rho c} + \frac{\rho c}{S}\right)\left(-\frac{i\rho c}{S_b}\right) + 2}}$$

$$\cancel{= \frac{-\frac{i2S}{S_b} + 4}{-\frac{iS}{S_b} - \frac{i(\rho c)^2}{S_b S} + 2}}$$

$$\Rightarrow \frac{A_2}{A_1} = \frac{\frac{2S}{\rho c} Z_0 + 4}{\frac{S}{\rho c} Z_0 + \frac{\rho c}{S} Z_0 + 2}$$

$$= \frac{\frac{2S}{\rho c}\left(\frac{-i\rho c}{S_b}\right)\cot kL + 4}{\frac{S}{\rho c}\left(\frac{-i\rho c}{S_b}\right)\cot kL + \left(\frac{\rho c}{S}\right)\left(\frac{-i\rho c}{S_b}\right)\cot kL}$$

$$= \frac{\frac{2S}{\rho c}\left(\frac{-i\rho c}{S_b}\right) + 4\tan kL}{\frac{S}{\rho c}\left(\frac{-i\rho c}{S_b}\right) + \left(\frac{\rho c}{S}\right)\left(\frac{-i\rho c}{S_b}\right)}$$

$$= \frac{\frac{-i2S}{S_b} + 4\tan kL}{-i\,S/S_b - \frac{i\rho^2c^2}{S\,S_b}}$$

$$= \frac{-i2S^2 + 4SS_b\tan kL}{-i(S^2 + \rho^2c^2)}$$

$$= \frac{2S^2 + i4SS_b\tan kL}{S^2 + \rho^2C^2}$$

$$\Rightarrow \left|\frac{A_2}{A_1}\right|^2 = \alpha_t = \frac{4S^4 + 16S^2S_b^2\tan^2 kL}{[S^2+\rho^2C^2]^2} \quad ?$$

$$\alpha_t = \frac{1}{1 + \left(\frac{S_b}{S}\right)^2 \tan^2 kL}$$

9)

8-2) $l_{eff} = \frac{16}{3\pi} a$

$\omega_0 = c\sqrt{\frac{S}{lV}}$

$S = \pi r^2 = \pi 10^{-2}$

$l' = \frac{16}{3\pi} 10^{-1}$ $+ l$

$V = .3 \times .5 \times .4 = .15 \times .4 = .06 m^3$ ✓

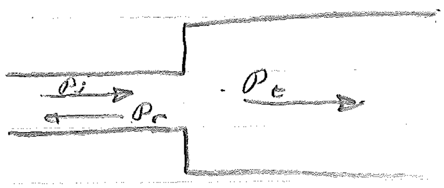
$C = 331$ 343

$\omega_0 = 331\sqrt{\frac{\pi 10^{-2}}{\frac{16}{3\pi} 10^{-1} (0.06)}}$

$f_0 = \frac{\omega_0}{2\pi} = 331 \left(\frac{1}{2}\right) \sqrt{\frac{10^{-2}}{\frac{16}{3} 10^{-1} (0.06)}}$

$= 29.4$ Hz

10)
(8-3)



a) $P_i = A_1 e^{i(\omega t - kx)}$

$P_r = B_1 e^{i(\omega t + kx)}$

$P_t = A_2 e^{i(\omega t - kx)}$

$P_L|_{x=0} = P_R|_{x=0} \Rightarrow A_1 + B_1 = A_2$

$U_L S_1|_{x=0} = U_R S_2|_{x=0} \Rightarrow \frac{1}{\rho c}[A_1 - B_1]S_1 = \frac{A_2}{\rho c} S_2$

$(A_1 - B_1)S_1 = A_2 S_2$

$$\therefore \frac{A_1 + B_1}{A_1 - B_1} = \frac{S_1}{S_2}$$

AND $\frac{B_1}{A_1} = \frac{S_1/S_2 - 1}{S_1/S_2 + 1}$ ✓

RATIO INTENSITY OF REFLECTED WAVE TO INCIDENT WAVE:

$$\left|\frac{B_1}{A_1}\right|^2 = \left(\frac{S_1 - S_2}{S_1 + S_2}\right)^2$$

NOW $A_1 + B_1 = A_2$

$1 + \frac{B_1}{A_1} = \frac{A_2}{A_1}$

$$\Rightarrow \frac{A_2}{A_1} = \frac{S_1 - S_2}{S_1 + S_2} + 1$$

$$= \frac{S_1 - \cancel{S_2} + S_1 + \cancel{S_2}}{S_1 + S_2}$$

$$= \frac{2S_1}{S_1 + S_2}$$

$$\left|\frac{A_2}{A_1}\right|^2 = \frac{4S_1^2}{(S_1 + S_2)^2}$$ ✓ ⇐ RATIO OF TRANSMITTED TO INCIDENT INTENSITY

b) $\left|\frac{A_2}{A_1}\right|^2 = \frac{4S_1^2}{(S_1+S_2)^2}$

TRANSMITTED WAVE WILL HAVE GREATER INTENSITY IF:

$$\frac{4S_1^2}{(S_1+S_2)^2} > 1$$

$$4S_1^2 > (S_1+S_2)^2$$

$$2S_1 > S_1+S_2 \quad ✓$$

$$S_1 > S_2$$

THE WAVE IS "SQUEEZED" INTO A SMALLED DIAMATER PIPE

c) $SWR = \frac{A_1+B_1}{A_1-B_1} = \frac{S_1}{S_2}$ FROM PART (a)

✓ fr $S_1 > S_2$

11)

8-13) a) $\omega_0 = c\sqrt{\frac{S}{l' V}}$

$V = \frac{S}{l_1}\left(\frac{c}{\omega_0}\right)^2$ ✓

$S = \pi r^2 = \pi(8\times10^{-2})^2$

$l_1 = \cancel{lA^0}\, l_{40}$

$= l_{eff}$

$= \frac{16 a}{3\pi} = \frac{16(8\times10^{-2})}{3\pi}$ ✓

$\omega_0 = 2\pi\times30$ ✓

$c = 343\ {}^{m}/\mathrm{sec}$ ✓

$$V = \frac{3\pi(8\times10^{-2})^{2}}{16(8\times10^{-2})}\left(\frac{3.43^2\times10^4}{4\pi^2\, 9\times10^2}\right)$$

$$= \frac{24\times10^{-2}}{16}\ \frac{(3.43)^2}{3.6}\times10^{2}$$

$$= \frac{4\ (3.43)^2}{16\times9}$$

.49

$$= \frac{11.8}{36} = \boxed{0.328}\ m^3$$

b)

$$\alpha_t = \left[1 + \frac{c^2}{4S^2(\omega l'/S - c^2/\omega V)^2}\right]^{-1}$$ ✓

$$= \left[1 + \frac{c^2}{4(\omega l' - Sc^2/\omega V)^2}\right]^{-1}$$

$$= \left[1 + \frac{(3.43)^2\times10^4}{4\left(\frac{2\cancel{\pi}\cdot 60\cdot 16\cdot 8\times10^{-2}}{3\cancel{\pi}} - \frac{\overset{32}{\cancel{\pi 64}}\times\cancel{10^{-4}}(3.43)^2\times\cancel{10^4}}{2\cancel{\pi}\, 60\times0.328}\right)^2}\right]^{-1}$$

$$= \left[1 + \frac{11.8\times10^4}{4\left(40\cdot16\cdot8\times10^{-2} - \frac{32(11.8)}{120\times3.28}\right)^2}\right]^{-1}$$

$$= \left[1 + \frac{11.8\times10^4}{4(51.2 - 9.6)^2}\right]^{-1}$$

51.2
-9

$$= \left[1 + \frac{11.8 \times 10^4}{4 (4.1.6)^2 \times 10^2} \right]^{-1}$$

$$= \left[1 + \frac{2\,9.5}{1.7.4} \right]^{-1}$$

$$= \left[1 + \overset{0.9}{\boxed{17}} \right]^{-1}$$

$$= \cancel{\frac{1}{18}} = \cancel{0.0555} \quad \boxed{.5}$$

12)

VI 7) $$c^2\left[\frac{\delta^2\rho}{\delta r^2}+\frac{1}{r}\frac{\delta\rho}{\delta r}+\frac{1}{r^2}\frac{\delta^2\rho}{\delta\phi^2}+\frac{\delta^2\rho}{\delta z^2}\right]=\frac{\delta^2\rho}{\delta t^2}$$

LET $\rho(r,\phi,z,t)=R(r)\,\Phi(\phi)\,Z(z)\,H(t)$

$$\begin{cases}\dfrac{\delta^2\rho}{\delta r^2}=\dfrac{d^2R(r)}{dr^2}\,\Phi(\phi)\,Z(z)\,H(t)\\ \dfrac{\delta\rho}{\delta r}=\dfrac{dR(r)}{dr}\,\Phi(\phi)\,Z(z)\,H(t)\\ \dfrac{\delta^2\rho}{\delta\phi^2}=R(r)\,\dfrac{d^2\Phi(\phi)}{d\phi^2}\,Z(z)\,H(t)\\ \dfrac{\delta^2\rho}{\delta z^2}=R(r)\,\Phi(\phi)\,\dfrac{d^2Z(z)}{dz^2}\,H(t)\\ \dfrac{\delta^2\rho}{\delta t^2}=R(r)\,\Phi(\phi)\,Z(z)\,\dfrac{d^2H(t)}{dt^2}\end{cases}$$

$$c^2\left[\frac{\delta^2R(r)}{dr^2}\Phi(\phi)Z(z)H(t)+\frac{1}{r}\frac{\delta R(r)}{\delta r}\Phi(\phi)Z(z)H(t)+\frac{1}{r^2}R(r)\frac{d^2\Phi(\phi)}{d\phi^2}Z(z)H(t)+R(r)\Phi(\phi)\frac{d^2Z(z)}{dz^2}H(t)\right]$$
$$=R(r)\,\Phi(\phi)\,Z(z)\frac{d^2H(t)}{dt^2}$$

$$c^2\left[\frac{1}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{1}{rR(r)}\frac{dR(r)}{dr}+\frac{1}{r^2\Phi(\phi)}\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}\right]$$
$$=\frac{1}{H(t)}\frac{d^2H(t)}{dt^2}=-\omega^2$$

$$\frac{1}{H(t)}\frac{d^2H(t)}{dt^2}=-\omega^2$$
$$\frac{d^2H(t)}{dt^2}=-\omega^2H(t)$$
$$\Rightarrow H(t)=A_3\cos\omega t+B_3\sin\omega t$$

$$c^2\left[\frac{1}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{1}{rR(r)}\frac{dR(r)}{dr}+\frac{1}{r^2\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}\right]=-\omega^2$$
$$c^2\left[\frac{1}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{1}{rR(r)}\frac{dR(r)}{dr}+\frac{1}{r^2\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}\right]=-\omega^2-\frac{c^2}{Z(z)}\frac{d^2Z(z)}{dz^2}$$
$$\frac{1}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{1}{rR(r)}\frac{dR(r)}{dr}+\frac{1}{r^2\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=-\left(\frac{\omega}{c}\right)^2-\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}$$
$$=\alpha^2-\left(\frac{\omega}{c}\right)^2=\text{CONSTANT}$$

$$\frac{-1}{Z(z)}\frac{d^2Z(z)}{dz^2}=\alpha^2$$
$$\frac{d^2Z(z)}{dz^2}=-\alpha^2Z(z)$$
$$\Rightarrow Z(z)=[A_2\cos\alpha z+B_2\sin\alpha z]$$

$$\frac{1}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{1}{rR(r)}\frac{dR(r)}{dr}+\frac{1}{r^2\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi}=\alpha^2-k^2\quad\ni k=\frac{\omega}{c}$$
$$\frac{1}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{1}{rR(r)}\frac{dR(r)}{dr}+(k^2-\alpha^2)=\frac{1}{r^2}\frac{1}{\Phi(\phi)}\frac{d^2\phi(\phi)}{d\phi^2}$$
$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr}+\frac{r}{R(r)}\frac{dR(r)}{dr}+r^2(k^2-\alpha^2)=-\{\Phi(\phi)\}^{-1}\frac{d^2\phi(\phi)}{d\phi^2}=+m^2$$
$$\frac{1}{\Phi(\phi)}\frac{d^2\phi(\phi)}{d\phi^2}=-m^2$$
$$\frac{d^2\phi(\phi)}{d\phi^2}=-m^2\,\phi(\phi)$$
$$\Rightarrow\phi(\phi)=A_1\cos m\phi+B_1\sin m\phi$$

LEAVING:

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + r^2(k^2-\alpha^2) = m^2$$

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left[k^2-\alpha^2-\left(\frac{m}{r}\right)^2\right]R(r)=0$$

LET $k_1^2 = k^2-\alpha^2$

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{R(r)}{dr} + \left[k_1-\left(\frac{m}{r}\right)^2\right]R(r)=0$$

EQ. 5.10 OF HAND OUT TEXT, WHICH IS SOLVED BY EXPANDING R(r) IN A PWR. SERIES. THE SOLUTION OF BESSEL'S EQUATION RESULTS IN THE BESSEL FUNCTIONS OF THE FIRST KIND. (EQ. 5.14)

$$\Rightarrow R(r) = J_m(k_1 r)$$
$$= J_m\left[(k^2-\alpha^2)^{\frac{1}{2}}r\right]$$

THEN:

$$\rho(r,\phi,z,t) = R(r)\,\Phi(\phi)\,Z(z)\,H(t)$$
$$= J_m\left[(k^2-\alpha^2)^{\frac{1}{2}}r\right]\left[A_1\cos m\phi + B_1\sin m\phi\right]$$
$$\left[A_2\cos\alpha z + B_2\sin\alpha z\right]\left[A_3\cos\omega t + B_3\sin\omega t\right]$$

✓

FINAL EXAM ACOUSTICS

Nov. 22, 1972

1. In the MKS system what units would the following quantities have?

   a) specific acoustic impedanse NT. METERS/SEC

   b) radiation impedance NT./SEC ?

   c) intensity at a point in a fluid WATTS/METER

   d) sound power transmission coefficient (NONE)

   e) shear modulus $\frac{\text{NEWTONS}}{\text{SQUARE METER}}$

   f) bending moment NEWTON-METERS

   g) the "Q" of a mechanical system. NONE (14)

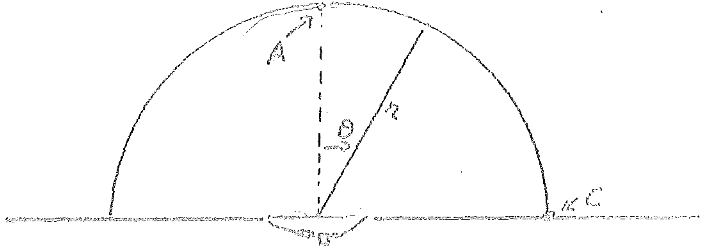
2. What is the fundamental frequency of a system consisting of air at 20° in a box of dimensions 0.1 m x 0.1 m x 2 meters? Indicate the location of any nodal planes in such a box if the gas is vibrating in its fundamental mode. (16)

3. Two basic equations are used in the derivation of the wave equations for waves in fluids.

   a) Write down these two equations.

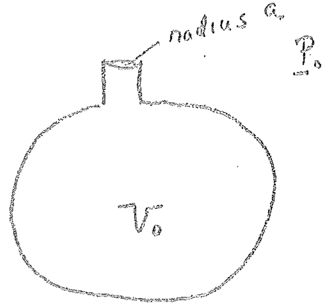
   b) Derive the wave equation in Cartesian coordinates for waves in a fluid. (20)

6. A loud speaker mounted in a large baffle is vibrating at a frequency of 3800 hertz.

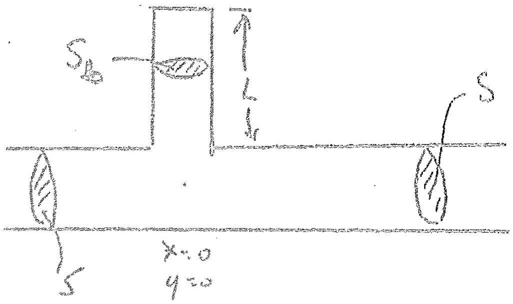


A microphone is placed at various points around a large semi-circle of radius 10 meters drawn with the center of the speaker as center, and the amplitude of the signal as noted. If the amplitude is largest at point A and falls off to zero at point C, find the diameter of the speaker, assuming it can be considered as a piston vibrating in an infinite babble. (25)

7. Derive an expression for the resonant frequency of an Helmholtz resonator when excited by an acoustic pressure $Ae^{i\omega t}$ at the top surface of the neck. Assume the mass of the gas in the neck moves as a unit, and that the air in the main container behaves as an ideal gas, the compressions and expansions of which take place adiabatically. (25)



8. Explain the procedure used to calculate the fraction of energy transmitted when a wave in a pipe encounters a branch tube. Calculate the sound power transmission coefficient for the case shown in the figure. (25)



Eigen functions for a fluid in a rectangular box of dimensions L_x, L_y, L_z

$$p(x,y,z,t) = \cos\frac{n_x\pi}{L_x}\cos\frac{n_y\pi}{L_y}\cos\frac{n_z\pi}{L_z}\left[A_{n_xn_yn_z}\cos\omega_{n_xn_yn_z}t + B_{n_xn_yn_z}\sin\omega_{n_xn_yn_z}t\right]$$

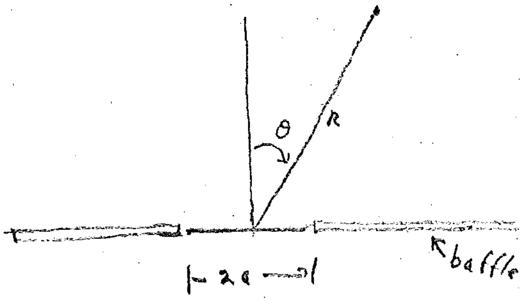
$$\omega_{n_xn_yn_z} = \pi c\sqrt{\left(\frac{n_x}{L_x}\right)^2+\left(\frac{n_y}{L_y}\right)^2+\left(\frac{n_z}{L_z}\right)^2}$$

$n_x = 0, 1, 2, 3 \ldots$

$n_y = 0, 1, 2, 3 \ldots$

$n_z = 0, 1, 2, 3 \ldots$

Radiation From Piston Vibrating in an infinite baffle



Point for which $\frac{r}{a} << 1$

$$\underset{\sim}{p} = \frac{i\rho c k U_0 a^2}{2\pi} e^{i(\omega t-kr)}\left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]$$

$$= \frac{i\rho c k U_0 a^2}{2\pi} e^{i(\omega t-kr)}\left[1 - \frac{(ka\sin\theta)^2}{8} + \frac{(ka\sin\theta)^4}{192} + \cdots\right]$$

For point on axis

$$\underset{\sim}{p} = -\rho c U_0 e^{i\omega t}\left[e^{-ik\sqrt{r^2+a^2}} - e^{ikr}\right]$$

Radiation Impedance of Piston Vibrating IN INFINITE BAFFLE

$$\underset{\sim}{Z_r} = \rho c\pi a^2\left[R_1(2ka) + i X_1(2ka)\right]$$

$$\cong \rho c\pi a^2\left[\frac{(ka)^2}{2} + i\frac{8}{3\pi}ka\right] \quad \text{if } ka << 1$$

Speed of sound in air 343 m/sec at 20°
Density of air $\rho = 1.21$ kg/m³ at 20°
$\Big\}$ $(\rho c)_{air} = 415$

Speed of sound in water 1.48×10^3 m/sec
Density of water 1.026×10^3 kg/m³
$\Big\}$ $\rho c = 1.48\times10^6$

GENERAL INFORMATION

EIGEN FUNCTIONS FOR MEMBRANE WITH RECTANGULAR BOUNDARY

$$z_{mn} = A_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \cos(\omega_{mn} t + \Omega_{mn})$$

$$\omega_{mn} = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \begin{array}{l} m = 1, 2, 3, \ldots\ldots \\ n = 1, 2, 3, \ldots\ldots \end{array}$$

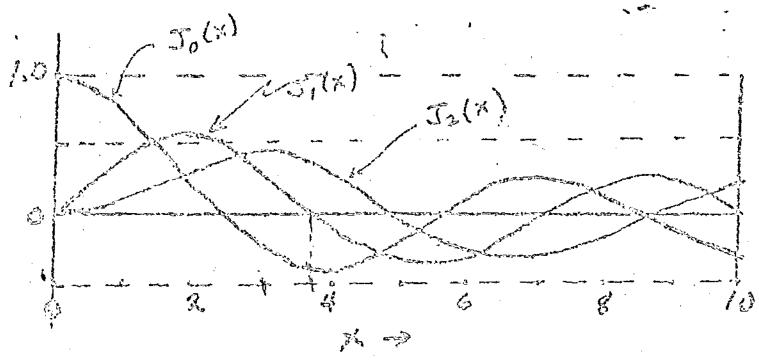
EIGEN FUNCTIONS FOR MEMBRANE WITH A CIRCULAR BOUNDARY

$$z_{mn} = A_{mn} J_m(k_n r) \cos(m\phi + \alpha_m) \cos(\omega_{mn} t + \Omega_{mn})$$

$$\omega_{01} = \frac{2.405}{a} c \quad ; \quad \omega_{02} = \frac{5.520}{a} c \quad ; \quad \omega_{03} = \frac{8.654}{a} c \;;$$

$$\omega_{11} = \frac{3.832}{a} c \quad ; \quad \omega_{12} = \frac{7.016}{a} c \quad ; \quad \omega_{13} = \frac{10.174}{a} c$$

$$\omega_{21} = \frac{5.136}{a} c \quad ; \quad \omega_{22} = \frac{8.417}{a} c \quad ; \quad \omega_{23} = \frac{11.620}{a} c$$



BOB MARK

1.) a) $\frac{\text{NT - METER}}{\text{SEC}}$

b) NT/SEC

c) WATTS/METER ✓

d) NONE ✓

e) $\frac{\text{NEWTONS}}{\text{SQUARE METER}}$ ✓

f) NEWTON METER ✓

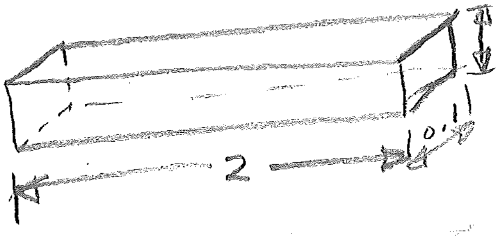
g) NONE ✓

1.	10
2	13
3	14
4	13
5	28
6	25
7	5
8	20
	128

.8
160) 128.0
128 0

2) $$\omega_{n_x n_y n_z} = \pi c \sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_y}{L_y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2}$$

$L_x = L_y = 0.1 \quad ; L_z = 2$



~~$\rho = (\cos_x \cos_y \cos_z)\left[A \cos \omega_{xyz} t + B \sin \omega_{n_x n_y n_z} t\right]$~~

~~$-\frac{d\Phi}{\delta t} = (\quad)\left[A \sin \omega t - B \sin \omega t\right]$~~

FUNDEMENTAL (LOWEST NON-ZERO) FREQUENCY:

$n_x = n_y = 0 \quad ; n_z = 1$

✓

$$\omega_{001} = \frac{\pi c}{L_z} \Rightarrow f_{001} = \frac{\omega_{001}}{2\pi} = \frac{c}{2L_z} = \frac{343}{4} = 85.8\,HZ$$

THE WALLS ARE THE ONLY NODAL PLANES ($\dot{X} = V = 0$)

How about pressure nodal planes?

3)a) $P = \overset{-}{\swarrow} B_a \nabla \cdot \vec{s}$

$-\nabla P = \rho_0 \frac{\delta^2 \vec{s}}{\delta t^2}$

a) $P = -B_a\left(\frac{\delta \xi}{\delta x} + \frac{\delta \eta}{\delta Y} + \frac{\delta \zeta}{\delta z}\right)$

$$\underline{\frac{\delta P}{\delta x}\hat{i} + \frac{\delta P}{\delta Y}\hat{j}} + \frac{\delta P}{\delta z}\hat{k} = \rho_0\left[\frac{\delta^2 \xi}{\delta t^2}\hat{i} + \frac{\delta^2 \eta}{\delta t^2}\hat{j} + \frac{\delta^2 \zeta}{\delta t^2}\hat{k}\right] \quad ?$$

$$\frac{\delta^2 P}{\delta t^2} = B_a\left[\frac{\delta}{\delta x}\frac{\delta^2 \xi}{\delta t^2} + \frac{\delta}{\delta Y}\frac{\delta^2 \eta}{\delta t^2} + \frac{\delta}{\delta z}\frac{\delta^2 \zeta}{\delta t^2}\right]$$

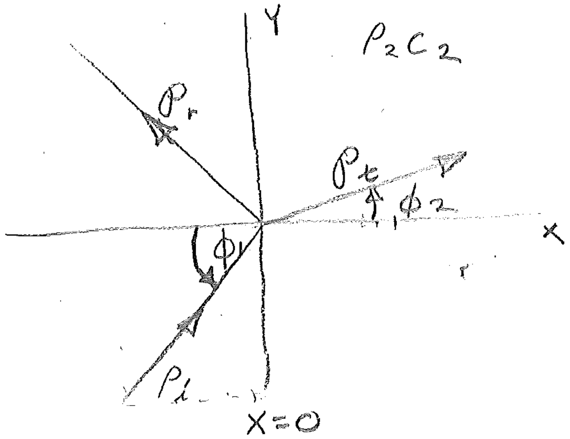
$$\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta Y^2} + \frac{\delta^2 P}{\delta z^2} = \rho_0\left[\frac{\delta}{\delta x}\frac{\delta^2 \xi}{\delta t^2} + \frac{\delta}{\delta Y}\frac{\delta^2 \eta}{\delta t^2} + \frac{\delta}{\delta z}\frac{\delta^2 \zeta}{\delta t^2}\right]$$

THUS:

$$\frac{1}{\rho_0}\left[\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta Y^2} + \frac{\delta^2 P}{\delta z^2}\right] = \frac{1}{B_a}\frac{\delta^2 P}{\delta t^2}$$

$$c^2\left[\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta Y^2} + \frac{\delta^2 P}{\delta z^2}\right] = \frac{\delta^2 P}{\delta t^2} \ni c = \sqrt{B_a/\rho_0}$$

4)



a) $P_i = A_1 e^{i(\omega t - x\cos\phi_1 - Y\sin\phi_1)} \Rightarrow U_i = \frac{A_1}{\rho_1 c_1} e^{i(\omega t - x\cos\phi_1 - Y\sin\phi_1}$ ✓

$P_r = B_1 e^{i(\omega t + x\cos\phi_1 - Y\sin\phi_1)} \Rightarrow U_R = \frac{-B_1}{\rho_1 c_1} e^{i(\text{———})}$

$P_t = A_2 e^{i(\omega t - x\cos\phi_2 - Y\sin\phi_2)} \Rightarrow U_t = \frac{A_2}{\rho_2 c_2} e^{i(\text{———})}$

WHERE: $\frac{\sin\phi_1}{c_1} = \frac{\sin\phi_2}{c_2}$ ✓

b) BOUNDRY CONDITIONS

$P_i\Big|_{x=0} \cos\phi_1 = P_r\Big|_{x=0} \cos\phi_2$?

$U_i\Big|_{x=0} \cos\phi_1 = U_r\Big|_{x=0} \cos\phi_2$?

c) $[P_i U_i]_x = [P_r U_r]_x + [P_t U_t]_x$?

WHERE ALL P'S & U'S ARE MAGNITUDES IN X DIR.

$$\frac{|A_1|^2}{\rho_1 c_1}\cos\phi_1 = \ominus\frac{|B_1|^2}{\rho_1 c_1}\cos\phi_1 + \frac{|A_2|^2}{\rho_2 c_2}\cos\phi_2$$

?

5) a) $P = A e^{i(\omega t - kx)} + B e^{i(\omega t + kx)}$

$$U = \frac{A}{\rho c} e^{i(\omega t - kx)} - \frac{B}{\rho c} e^{i(\omega t + kx)}$$

$$U|_{x=0} = 0 = \frac{A}{\rho c} e^{i\omega t} - \frac{B}{\rho c} e^{i\omega t}$$

$$0 = A - B$$

$$\therefore \underset{\sim}{A} = \underset{\sim}{B} \quad \checkmark$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$
$$e^{j\theta} - e^{-j\theta} = j2\sin\theta$$
$$\Rightarrow \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{j2}$$

b) $Z_{-L} = \dfrac{P|_{x=-L}}{U|_{x=-L}\, S}$

$$P = A e^{i\omega t}\left[e^{-ikx} + e^{i\omega t}\right]$$

$$= 2A e^{i\omega t} \cos kx$$

$$U = \frac{-A}{\rho c} e^{i\omega t}\left[e^{ikx} - e^{-ikx}\right]$$

$$= \frac{-i2A}{\rho c} e^{i\omega t} \sin kx$$

$$\Rightarrow Z_{-L} = \frac{\cancel{2A}\cancel{e^{i\omega t}} \cos kL}{\frac{-i\cancel{2A}}{\rho c}\cancel{e^{i\omega t}} \sin kL}$$

$$= \frac{i\rho c}{S} \cot kL$$

c) RESONANCE WHEN $\cot kL = 0$

$$\Rightarrow \cos kL = 0$$

$$\cos \frac{\omega}{c} L = 0$$

THUS $\dfrac{\omega}{c} L = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \ldots, \dfrac{(2n+1)\pi}{2}, \ldots$

$$\omega_n = \frac{(2n+1)\pi c}{2L}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{(2n+1)c}{4L} \quad \checkmark$$

$$6)\ P(r,\theta) = \frac{i\rho c k U_0 a^2}{2\pi} e^{i(\omega t - kr)} \left[\frac{2 J_1(ka \sin\theta)}{ka \sin\theta} \right]$$

$$P\left(r, \frac{\pi}{2}\right) = 0 = \frac{i\rho c k U_0 a^2}{2\pi} e^{i(\omega t - kr)} \left(\frac{2 J_1(ka)}{ka} \right)$$

$$P_A = \frac{\rho c k U_0 a^2}{2\pi} \left[\frac{2 J_1(ka \sin\theta)}{ka \sin\theta} \right]$$

$$P_A\left(\frac{\pi}{2}\right) = \frac{\rho c k U_0 a^2}{2\pi} \frac{2 J_1(ka)}{ka} = 0$$

$$\frac{2 J_1(ka)}{ka} = 0$$

THERE'S ONLY 1 ZERO MEASURED

SO $(ka) \cong 3.6 \leftarrow$ (FROM GRAPH FIRST (0) OF $J_1(x)/x$)

AND: $\underline{3.83}$

$$a \cong \frac{3.6}{k}$$

$$= \frac{3.6\, c}{\omega}$$

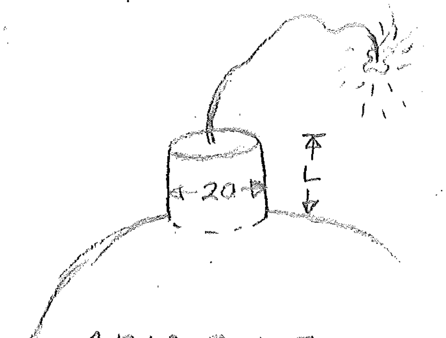
$$= \frac{(3.6)(3.43)\times\cancel{10^3}}{2\pi\ 3.8 \times \cancel{10^3}}$$

$$= \frac{123.5 \times 10^{-2}}{23.9}$$

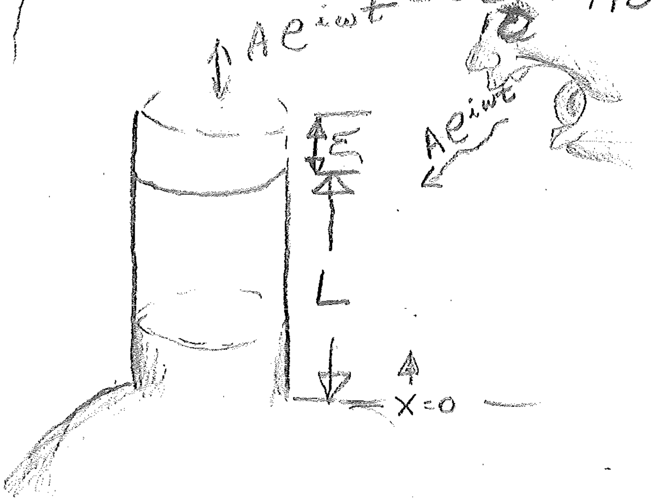
$$\cong 5.18 \text{ cm}$$

$$d = 2a \cong 10.36 \text{ cm}$$ ✓

7)



ADIABATIC BULK MODULUS $= \gamma P_0$



$C = \frac{V}{\rho c^2}$; $\cancel{M = \frac{c\ell'}{q}}$

$$\underbrace{\sigma L \pi a^2}_{\text{GAS MASS}}$$

MASS OF GAS TURNS OUT TO NOT BE BIG ENOUGH, SO ADD $\boxed{\rho c \pi a^2 R_1(2ka) = m_{eff}}$

$$\ell_{eff} = \rho c \pi a^2 X_1(2ka)$$

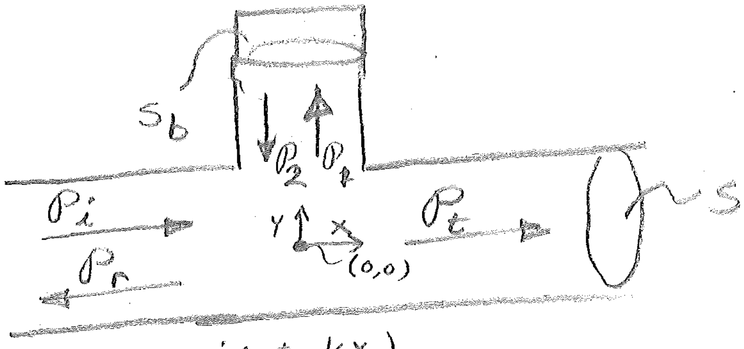
$$\ell' = L + \ell_{eff}$$

$$m' = \sigma L \pi a^2 + m_{eff}$$

RESONANCE OCCURS WHEN THE NEW IMPEDANCE IS MINIMUM:

$$\omega_0 = \frac{1}{\sqrt{MC}}$$

8)



$P_i = A_1 e^{i(\omega t - kx)}$
$P_r = B_1 e^{i(\omega t + kx)}$
$P_t = A_2 e^{i(\omega t - kx)}$ (Pg 8-a)
$P_1 = C_1 e^{i(\omega t - kY)}$
$P_2 = C_2 e^{i(\omega t + kY)}$

$P_r|_{x=0} = P_L|_{x=0} = P_U|_{Y=0} \Rightarrow A_1 + B_1 = A_2$

$S\,U_L|_{x=0} = S U_R|_{x=0} + S_b U_{UP}|_{Y=0}$

$$\frac{1}{P_L/SU_L} = \frac{1}{P_R/S_1 U_r} + \frac{1}{P_{UP}/S_b U_{UP}}$$

$$\frac{1}{Z_L} = \frac{1}{Z_R} + \frac{1}{Z_{UP}}$$

$$Z_L = \frac{Z_r Z_{UP}}{Z_r + Z_{UP}}$$

$$\frac{P_t}{SU_t} = \frac{B_1 \cancel{e^{i(\sim)}}}{\frac{S}{\rho c} B_1 \cancel{e^{i(\sim)}}} = \frac{\rho c}{S}$$

$$Z_L = \frac{\frac{\rho c}{S} Z_{UP}}{\frac{\rho c}{S} + Z_{UP}}$$ ✓

$P_{UP} = C_1 e^{i(\omega t - kY)} + C_2 e^{i(\omega t + kY)}$

$U_{UP} = \frac{C_1}{\rho c} e^{i(\sim)} - \frac{C_2}{\rho c} e^{i\sim)}$

TO
NEXT
PAGE

$$U_{up}\Big|_{Y=L} = 0 = \frac{c_1}{\rho c} e^{i(\omega t - kL)} - \frac{c_2}{\rho c} e^{i(\omega t + kY)}$$

$$\Rightarrow c_1 e^{-ikL} = c_2 e^{ikL}$$

$$c_1 = c_2 e^{i2kL} \checkmark$$

$$P_{up} = c_2\left[e^{i2kL} e^{i(\omega t - kY)} + e^{i(\omega t + kY)}\right]$$

$$U_p = \frac{c_2}{\rho c}\left[e^{i2kL} e^{i(\omega t - kY)} - e^{i(\omega t + kY)}\right)$$

so $$Z_{up} = \frac{P_{up}}{S_b U_{up}}\Big|_{Y=0}$$

$$= \frac{\rho c}{S_b} \frac{e^{i2kL} + 1}{e^{i2kL} - 1}$$

$$= \frac{\rho c}{i S_b} \frac{\left[e^{ikL} + e^{-ikL}\right]/2}{\left[e^{ikL} - e^{-ikL}\right]/i2}$$

$$= \frac{\rho c}{i S_b} \cot kL \checkmark$$

Pg (8-b)

so:
$$Z_L = \frac{\frac{\rho c}{S}\left[\frac{\rho c}{i S_b} \cot kL\right]}{\frac{\rho c}{S} + \left[\frac{\rho c}{i S_b} \cot kL\right]}$$

$$= \frac{(\rho c/S)(\rho c/i S_b)}{\frac{\rho c}{S} \tan kL + \frac{\rho c}{i S_b}}$$

NOW $$Z_L = \frac{P_L}{S U_L}\Big|_{x=0}$$

$$= \frac{\rho c(A_1 + B_1)}{S(A_1 - B_1)} = \frac{(\rho c/S)(-i\rho c/S_b)}{\frac{\rho c}{S} \tan kL + \rho c/i S_b}$$

(COVER)

$$\frac{P_{up}}{SU_{up}}\Big|_{x=0} \cancel{= \frac{C_1+C_2}{\frac{1}{\rho c}(C_1-C_2)} = \frac{1}{\rho c}\left(\frac{C_1+C_2}{C_1-C_2}\right)}$$

$$\frac{\cancel{\rho c}(A_1+B_1)}{S(A_1-B_1)} = \frac{(\cancel{\rho c/S})(-i\rho c/S_b)}{\frac{\rho c}{S}\tan kL - i\frac{\rho c}{S_b}}$$

$$\frac{A_1+B_1}{A_1-B_1} = \frac{-i\rho c S}{\rho c S_b \tan kL - \rho c S}$$

$$(A_1+B_1)(\rho c S_b \tan kL - \rho c S) = (A_1-B_1)(-i\rho c S)$$

$$B_1\left[\rho c S_b \tan kL - \rho c S - i\rho c S\right]$$

$$= A_1\left[-\rho c S_b \tan kL + \rho c S \quad - i\rho c S\right]$$

$$\frac{B_1}{A_1} = \frac{-\rho c S_b \tan kL + \rho c S(j-1)}{\rho c S_b \tan kL - \rho c S(j+1)}$$

NOW: $A_1 + B_1 = A_2$

$1 + \frac{B_1}{A_1} = \frac{A_2}{A_1}$

$$\frac{A_2}{A_1} = \frac{-\rho c S_b \tan kL - \rho c S(j-1) + \rho c S_b \tan kL - \rho c S(j+1)}{\rho c S_b \tan kL - \rho c S(j+1)}$$

$$= \frac{-2\cancel{\rho c}S_b \tan kL - j2\cancel{\rho c}S}{\cancel{\rho c}S_b \tan kL - \cancel{\rho c}S - j\cancel{\rho c}S}$$

To Pg 8 d

(BACK OF SHEET ON WHICH THE WHOLE MESS STARTED)

(Pg 8-c)

$$U_{up}\Big|_{Y=L} = 0 = \frac{C_1}{\rho c} e^{i(\omega t - kL)} - \frac{C_2}{\rho c} e^{i(\omega t + kY)}$$

$$\Rightarrow C_1 e^{-ikL} = C_2 e^{ikL}$$

$$C_1 = C_2 e^{i2kL} \checkmark$$

$$P_{up} = C_2\left[e^{i2kL} e^{i(\omega t - kY)} + e^{i(\omega t + kY)}\right]$$

$$U_p = \frac{C_2}{\rho c}\left[e^{i2kL} e^{i(\omega t - kY)} - e^{i(\omega t + kY)}\right)$$

so $$Z_{up} = \frac{P_{up}}{S_b U_{up}}\Big|_{Y=0}$$

$$= \frac{\rho c}{S_b} \frac{e^{i2kL} + 1}{e^{i2kL} - 1}$$

$$= \frac{\rho c}{iS_b} \frac{\left[e^{ikL} + e^{-ikL}\right]/2}{\left[e^{ikL} - e^{-ikL}\right]/i2}$$

$$= \frac{\rho c}{iS_b} \cot kL \checkmark$$

Pg (8-b)

so:
$$Z_L = \frac{\frac{\rho c}{S}\left[\frac{\rho c}{iS_b}\cot kL\right]}{\frac{\rho c}{S} + \left[\frac{\rho c}{iS_b}\cot kL\right]}$$

$$= \frac{(\rho c/S)(\rho c/iS_b)}{\frac{\rho c}{S}\tan kL + \frac{\rho c}{iS_b}}$$

NOW $$Z_L = \frac{P_L}{S U_L}\Big|_{x=0}$$

$$= \frac{\rho c(A_1 + B_1)}{S(A_1 - B_1)} = \frac{(\rho c/S)(-i\rho c/S_b)}{\frac{\rho c}{S}\tan kL + \rho c/iS_b}$$

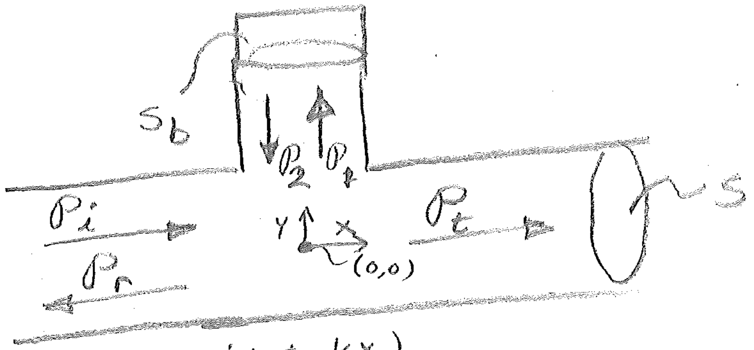
(OVER)

(Pg 8d)

$\left|\frac{A_2}{A_1}\right|^2$ = TRANSMISSION PWR. COEFFICIENT)

$$\left|\frac{A_2}{A_1}\right|^2 = \frac{4 S_b^2 \tan^2 kL + 4 S^2}{(S_b^2 \tan^2 kL - S)^2 + S^2}$$

8)



$P_i = A_1 e^{i(\omega t - kx)}$

$P_r = B_1 e^{i(\omega t + kx)}$

$P_t = A_2 e^{i(\omega t - kx)}$ (Pg 8-a)

$P_1 = C_1 e^{i(\omega t - kY)}$

$P_2 = C_2 e^{i(\omega t + kY)}$

$$P_r|_{x=0} = P_L|_{x=0} = P_U|_{Y=0} \Rightarrow A_1 + B_1 = A_2$$

$$S U_L|_{x=0} = S U_R|_{x=0} + S_b U_{UP}|_{Y=0}$$

$$\frac{1}{P_L/SU_L} = \frac{1}{P_R/S_1 U_r} + \frac{1}{P_{UP}/S_b U_{UP}}$$

$$\frac{1}{Z_L} = \frac{1}{Z_R} + \frac{1}{Z_{UP}}$$

$$Z_L = \frac{Z_r Z_{UP}}{Z_r + Z_{UP}}$$

$$\frac{P_t}{SU_t} = \frac{\cancel{B_1 e^{i(\sim)}}}{\frac{S}{\rho c}\cancel{B_1 e^{i(\sim)}}} = \frac{\rho c}{S}$$

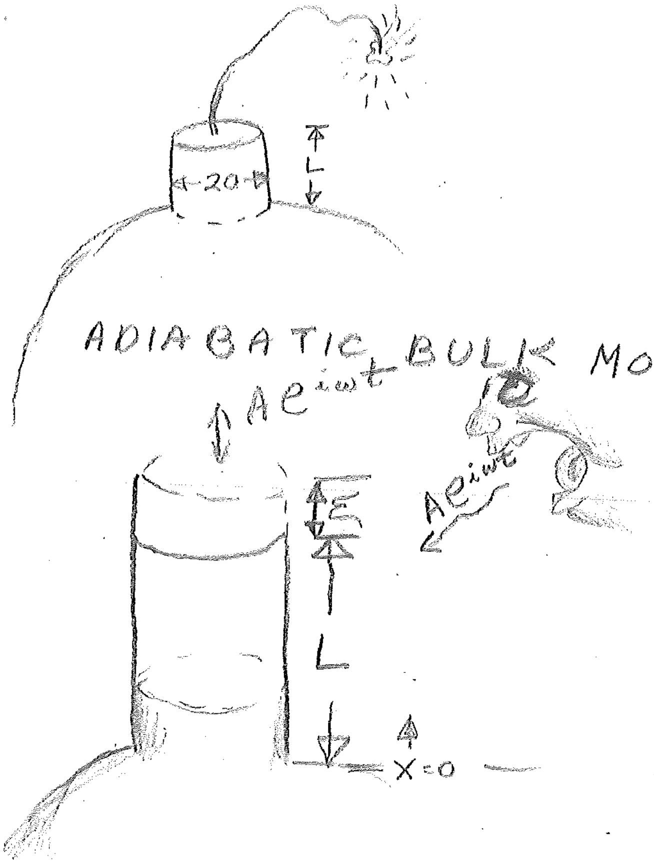
$$Z_L = \frac{\frac{\rho c}{S} Z_{UP}}{\frac{\rho c}{S} + Z_{UP}} \checkmark$$

$$P_{UP} = C_1 e^{i(\omega t - kY)} + C_2 e^{i(\omega t + kY)}$$

$$U_{UP} = \frac{C_1}{\rho c} e^{i(\sim)} - \frac{C_2}{\rho c} e^{i\sim}$$

TO NEXT PAGE

7)



ADIABATIC BULK MODULUS = γP_0

$$C = \frac{V}{\rho c^2} \; ; \; \cancel{M = \frac{c\ell}{g}}$$

$$\underbrace{\sigma L \pi a^2}_{\text{GAS MASS}}$$

MASS OF GAS TURNS OUT TO NOT BE BIG ENOUGH, SO ADD $\left(\rho c \pi a^2 R_1(2ka) = m_{eff}\right)$

$$\ell_{eff} = \rho c \pi a^2 X_1(2ka)$$

$$\ell' = L + \ell_{eff}$$

$$m' = \sigma L \pi a^2 + m_{eff}$$

RESONANCE OCCURS WHEN THE NEW IMPEDANCE IS MINIMUM:

$$\omega_0 = \frac{1}{\sqrt{MC}}$$

$$6)\ P(r,\theta) \cong \frac{i\rho c k U_0 a^2}{2\pi} e^{i(\omega t - kr)} \left[\frac{2 J_1(ka \sin\theta)}{ka \sin\theta}\right]$$

$$P(r,\tfrac{\pi}{2}) = 0 = \frac{i\rho c k U_0 a^2}{2\pi} e^{i(\omega t - kr)} \left(\frac{2 J_1(ka)}{ka}\right)$$

$$P_A = \frac{\rho c k U_0 a^2}{2\pi} \left[\frac{2 J_1(ka \sin\theta)}{ka \sin\theta}\right]$$

$$P_A(\tfrac{\pi}{2}) = \frac{\rho c k U_0 a^2}{2\pi} \; \frac{2 J_1(ka)}{ka} = 0$$

$$\frac{2 J_1(ka)}{ka} = 0$$

THERE'S ONLY 1 ZERO MEASURED

SO $(ka) \cong 3.6 \leftarrow$ (FROM GRAPH)
FIRST (0) OF $J_1(x)/x$

AND: $\underline{3.83}$

$$a \cong \frac{3.6}{k}$$

$$= \frac{3.6\,c}{\omega}$$

$$= \frac{(3.6)(3.43) \times 10^3}{2\pi\ 3.8 \times 10^3}$$

$$= \frac{1.23.5 \times 10^{-2}}{23.9}$$

$$\cong 5.18 \text{ cm}$$

$$d = 2a \cong 10.36 \text{ cm}$$

5) a) $P = Ae^{i(\omega t - kx)} + Be^{i(\omega t + kx)}$

$$U = \frac{A}{\rho c} e^{i(\omega t - kx)} - \frac{B}{\rho c} e^{i(\omega t + kx)}$$

$$U|_{x=0} = 0 = \frac{A}{\rho c} e^{i\omega t} - \frac{B}{\rho c} e^{i\omega t}$$

$$0 = A - B$$

$$\therefore \underset{\sim}{A} = \underset{\sim}{B}$$ ✓

$e^{j\theta} = \cos\theta + j\sin\theta$
$e^{-j\theta} = \cos\theta - j\sin\theta$
$e^{j\theta} - e^{-j\theta} = j2\sin\theta$
$\Rightarrow \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{j2}$

b) $$Z_{-L} = \frac{P|_{x=-L}}{U|_{x=-L}\, S}$$

$$P = Ae^{i\omega t}\left[e^{-ikx} + e^{i\omega t}\right]$$

$$= 2Ae^{i\omega t}\cos kx$$

$$U = \frac{-A}{\rho c} e^{i\omega t}\left[e^{ikx} - e^{-ikx}\right]$$

$$= \frac{-i2A}{\rho c} e^{i\omega t}\sin kx$$

$$\Rightarrow Z_{-L} = \frac{2A\cancel{e^{i\omega t}}\cos kL}{\frac{-i2A}{\rho c}\cancel{e^{i\omega t}}\sin kL}$$

$$= \underset{\uparrow}{i}\frac{\rho c}{S}\cot kL$$

c) RESONANCE WHEN $\cot kL = 0$

$$\Rightarrow \cos kL = 0$$

$$\cos \frac{\omega}{c} L = 0$$

THUS $\frac{\omega}{c} L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2n+1)\pi}{2}, \ldots$

$$\omega_n = \frac{(2n+1)\pi c}{2L}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{(2n+1)c}{4L}$$ ✓

GENERAL INFORMATION

EIGEN FUNCTIONS FOR MEMBRANE WITH RECTANGULAR BOUNDARY

$$z_{mn} = A_{mn} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \cos\left(\omega_{mn} t + \Omega_{mn}\right)$$

$$\omega_{mn} = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \begin{array}{l} m = 1, 2, 3, \ldots\ldots \\ n = 1, 2, 3, \ldots\ldots \end{array}$$

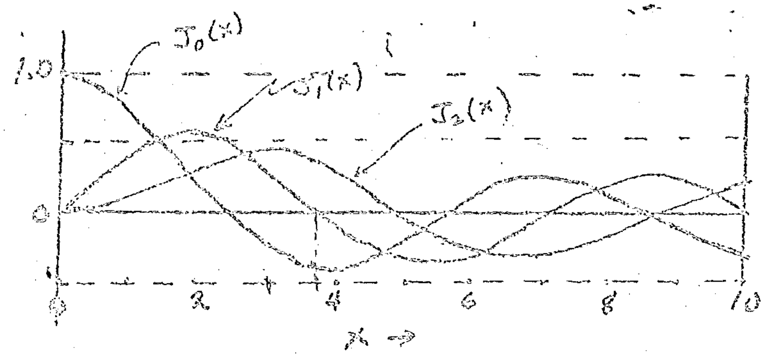
EIGEN FUNCTIONS FOR MEMBRANE WITH A CIRCULAR BOUNDARY

$$z_{mn} = A_{mn} J_m(k_n r) \cos(m\phi + \alpha_m) \cos(\omega_{mn} t + \Omega_{mn})$$

$$\omega_{01} = \frac{2.405}{a} c \quad ; \quad \omega_{02} = \frac{5.520}{a} c \quad ; \quad \omega_{03} = \frac{8.654}{a} c \ ;$$

$$\omega_{11} = \frac{3.832}{a} c \quad ; \quad \omega_{12} = \frac{7.016}{a} c \quad ; \quad \omega_{13} = \frac{10.174}{a} c$$

$$\omega_{21} = \frac{5.136}{a} c \quad ; \quad \omega_{22} = \frac{8.417}{a} c \quad ; \quad \omega_{23} = \frac{11.520}{a} c$$



BOB MARKS

1.) a) $\frac{\text{NT-METER}}{\text{SEC}}$

b) NT/SEC

c) WATTS/METER ✓

d) NONE ✓

e) $\frac{\text{NEWTONS}}{\text{SQUARE METER}}$ ✓

f) NEWTON METER ✓

g) NONE ✓

1.	10
2	13
3	14
4	13
5	28
6	25
7	5
8	20
	128

.8

160) 128.0

128 0

(1) c) $I = \frac{1}{\tau}\int_0^\tau F v\,dt$

$= (Ma)\left(\frac{L}{T}\right)$

$F = k\dot{x}$ $\dot{x} =$

FORCE = k $\frac{M}{SEC}$

$\rho c = \frac{M}{L^3}\frac{L}{SEC} = \frac{M}{L^2 SEC}$

mk = $\frac{\text{FORCE} \cdot \text{MASS}}{\text{VELOCITY}}$

$\frac{M\ell}{L^2 SEC}$

$\frac{M^2 \cancel{LT}}{T^2 \cancel{L}}$ M/NT

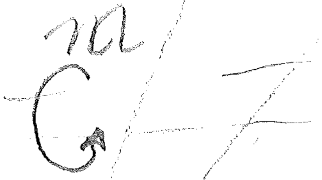
$\frac{ML^2}{SEC}$ $\frac{}{L^4}$

$\frac{F}{V}$

$G = \frac{S_{YZ}}{\theta} = \frac{1}{Y}\frac{F}{A\theta}$

KG /

$\epsilon_{xx} = \frac{1}{Y}S_{YX} = \frac{F}{A}$



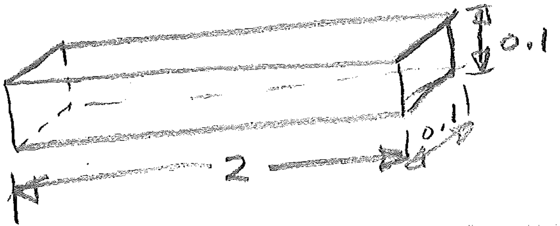
$F = Ev$

$\frac{FM}{SEC}$

$\tau = F \times X$

2) $$\omega_{n_x n_Y n_z} = \pi c \sqrt{\left(\frac{n_x}{L_x}\right)^2 + \left(\frac{n_Y}{L_Y}\right)^2 + \left(\frac{n_z}{L_z}\right)^2}$$

$L_x = L_Y = 0.1 \quad ; L_z = 2$



~~$\rho = (\cos_x \cos_Y \cos_z)\left[A \cos\omega_{n_x n_Y n_z} t + B \sin\omega t\right]$~~

~~$-\frac{d\Phi}{\delta t} = (\qquad)\left[A \sin\omega t - B \sin\omega t\right]$~~

FUNDEMENTAL (LOWEST NON-ZERO) FREQUENCY:

$n_x = n_Y = 0 \quad ; n_z = 1$

✓

$$\omega_{001} = \frac{\pi c}{L_z} \Rightarrow f_{001} = \frac{\omega_{001}}{2\pi} = \frac{c}{2L_z} = \frac{343}{4} = 85.8 \text{ HZ}$$

THE WALLS ARE THE ONLY NODAL PLANES $(X = V = 0)$

How about pressure nodal planes?

3)a) $P \overset{-}{=} B_a \nabla \cdot \vec{s}$

$-\nabla P = \rho_0 \dfrac{\delta^2 \vec{s}}{\delta t^2}$

a) $P = -B_a\left(\dfrac{\delta \xi}{\delta x} + \dfrac{\delta \eta}{\delta y} + \dfrac{\delta \zeta}{\delta z}\right)$?

$$\frac{\delta P}{\delta x}\hat{i} + \frac{\delta P}{\delta y}\hat{j} + \frac{\delta P}{\delta z}\hat{k} = \rho_0\left[\frac{\delta^2 \xi}{\delta t^2}\hat{i} + \frac{\delta^2 \eta}{\delta t^2}\hat{j} + \frac{\delta^2 \zeta}{\delta t^2}\hat{k}\right]$$

$$\frac{\delta^2 P}{\delta t^2} = B_a\left[\frac{\delta}{\delta x}\,\frac{\delta^2 \xi}{\delta t^2} + \frac{\delta}{\delta y}\,\frac{\delta^2 \eta}{\delta t^2} + \frac{\delta}{\delta z}\,\frac{\delta^2 \zeta}{\delta t^2}\right]$$

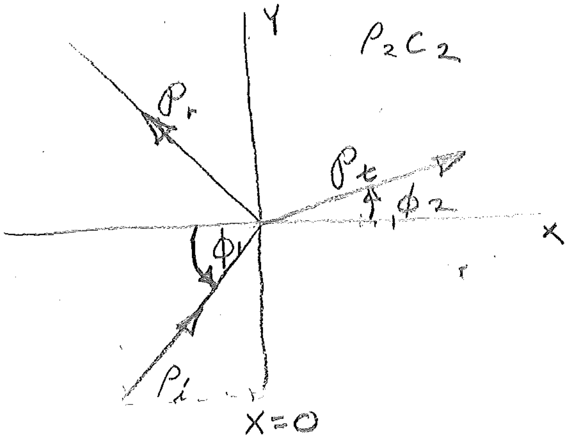
$$\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta y^2} + \frac{\delta^2 P}{\delta z^2} = \rho_0\left[\frac{\delta}{\delta x}\,\frac{\delta^2 \xi}{\delta t^2} + \frac{\delta}{\delta y}\,\frac{\delta^2 \eta}{\delta t^2} + \frac{\delta}{\delta z}\,\frac{\delta^2 \zeta}{\delta t^2}\right]$$

THUS:

$$\frac{1}{\rho_0}\left[\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta y^2} + \frac{\delta^2 P}{\delta z^2}\right] = \frac{1}{B_a}\,\frac{\delta^2 P}{\delta t^2}$$

$$c^2\left[\frac{\delta^2 P}{\delta x^2} + \frac{\delta^2 P}{\delta y^2} + \frac{\delta^2 P}{\delta z^2}\right] = \frac{\delta^2 P}{\delta t^2} \Rightarrow c = \sqrt{B_a/\rho_0}$$

4)



a) $P_i = A_1 e^{i(\omega t - x\cos\phi_1 - Y\sin\phi_1)} \Rightarrow U_i = \frac{A_1}{\rho_1 c_1} e^{i(\omega t - x\cos\phi_1 - Y\sin\phi_1)}$ ✓

$P_r = B_1 e^{i(\omega t + x\cos\phi_1 - Y\sin\phi_1)} \Rightarrow U_R = \frac{-B_1}{\rho_1 c_1} e^{i(\quad)}$

$P_t = A_2 e^{i(\omega t - x\cos\phi_2 - Y\sin\phi_2)} \Rightarrow U_t = \frac{A_2}{\rho_2 c_2} e^{i(\quad)}$

WHERE: $\frac{\sin\phi_1}{c_1} = \frac{\sin\phi_2}{c_2}$ ✓

b) BOUNDRY CONDITIONS

$P_i\big|_{x=0} \cos\phi_1 = P_r\big|_{x=0} \cos\phi_2$?

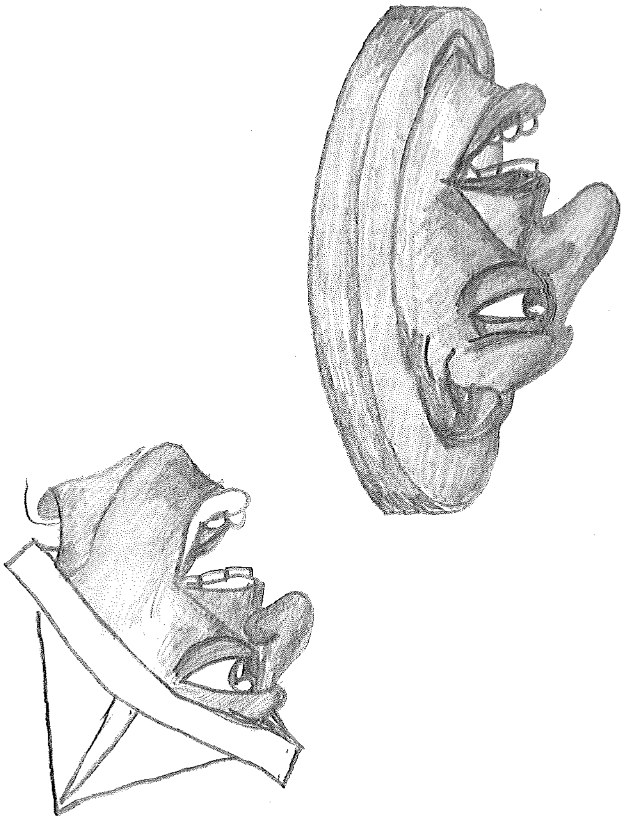
$U_i\big|_{x=0} \cos\phi_1 = U_r\big|_{x=0} \cos\phi_2$?

c) $[P_i U_i]_x = [P_r U_r]_x + [P_t U_t]_x$?

WHERE ALL P'S & U'S ARE MAGNITUDES IN X DIR.

$$\frac{|A_1|^2}{\rho_1 c_1}\cos\phi_1 = -\frac{|B_1|^2}{\rho_1 c_1}\cos\phi_1 + \frac{|A_2|^2}{\rho_2 c_2}\cos\phi_2$$

?



Sound and Vibration

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KINSLER & FREY

Dates		Problems	Due
Dec. 5 - 14	Chap. 10	Prob. 1, 4, 8, 11, 13, 16, 19, 22	Dec. 14
14 - 23	11	Prob. 2, 3, 4, 6, 9, 11, 14	23
Jan. 8 - 17	12	Prob. 1, 3, 8, 13, 16, 19, 23	Jan. 17
17 - 23	13	Prob. 1, 2, 5, 7, 9, 10	Jan. 23
23 - Feb. 1	14	Prob. 1, 3, 4, 6, 8, 14, 19, 21	Feb. 1
Feb. 1 - Feb. 9	15	Prob. 1, 2, 6, 7, 10, 17, 20, 25	Feb. 9
9 - 27	Finish paper		27

BOB MARKS
10-14-72
CHAPTER 10

10-1) a) $\eta = \dfrac{\phi^2 R_r}{\phi^2 (R_r + R_m) + R_E Z_m^2}$

EXCEPT IN THE IMMEDIATE VICINITY OF RESONANCE:

$$R_E Z_m^2 >> \phi^2 (R_r + R_m)$$ ✓

LEAVING:

$$\eta = \frac{\phi^2 R_r}{R_E \; Z_m^2}$$

FOR HIGH FREQUENCIES, $R_1(x) \sim 1$ AND $X_1(x) \sim 0$

$$\Rightarrow Z_m = Z_r + Z_c$$
$$= (R_r + R_m) + j\left[\omega m - \frac{s}{\omega} + X_r\right]$$
$$= \rho_0 c \pi a^2 R_1 (2 \tfrac{\omega a}{c}) + R_m + j\left[\omega m - \frac{s}{\omega} + \rho_0 c \pi a^2 X_1 (\tfrac{2 \omega a}{c})\right]$$
$$\simeq \rho_0 c \pi a^2 + R_m + j\left[\omega m - \frac{s}{\omega}\right]$$

FOR HIGH ENOUGH FREQUENCY, THE SYSTEM IS MASS CONTROLLED,

$$Z_m \simeq \rho_0 c \pi a^2 + R_m + j[\omega m]$$

AND THE FREQUENCY PROPORTIONED REACTANCE BECOMES LARGE WITH RESPECT TO THE CONSTANT RESISTANCE: ✓

$$Z_m \simeq j \omega m$$
$$Z_m^2 = (\omega m)^2$$

SUBSTITUTING:

$$\eta = \frac{\phi^2 R_r}{R_E \omega^2 m^2}$$ ✓

10.1 ✓
10.4 .90
10.8 .85
10.11 .9
10.13 ✓
10.16 .85
10.19 ✓
10.22 .30

6.8
8

b) $f = 10^3$

i) FOR $\eta = \dfrac{\phi^2 R_r}{\phi^2(R_r + R_m) + R_E Z_m^2}$

$$R_r = 13\, R_1 (3.66 \times 10^{-3} f)$$
$$= 13\, R_1 (3.66)$$
$$= (13)(0.961) = 12.5$$
$$X_r = 13\, X_1 (3.66 \times 10^{-3} f)$$
$$= 13\, X_1 (3.66)$$
$$= 13 (0.601)$$
$$= 7.81$$
$$Z_m = Z_r + Z_c$$
$$= (R_r + R_m) + j\left(\omega m - \frac{s}{\omega} + X_r\right)$$
$$= (12.5 + 1) + j\left(2\pi \times 10^3 \times 10^{-2} + 7.81 - \frac{2 \times 10^3}{2\pi \times 10^3}\right)$$
$$= 13.5 + j(62.8 + 7.81 - 0.32)$$
$$= 13.5 + j(70.3)$$
$$Z_m^2 = 1.35^2 \times 10^2 + 7.03^2 \times 10^2$$
$$= (1.82 + 49.5) \times 10^2$$
$$= 51.3 \times 10^2$$
$$\phi = 4.5$$
$$\phi^2 = 20.2$$
$$\eta = \frac{(20.2)(12.5)}{20.2(12.5 + 1) + 5 \times 5.13 \times 10^3}$$
$$= \frac{2.53 \times 10^2}{2.73 \times 10^2 + 256 \times 10^2}$$
$$= \frac{2.53}{2.59 \times 10^2}$$
$$= .977 \times 10^{-2}$$
$$= 0.977\%$$ ✓

ii) FOR $\eta = \frac{\phi^2 R_F}{m^2 \omega^2 R_E}$

$$\eta = \frac{(20.2)(42.5)}{10^{-4}(2\pi \times 10^3)^2 \, 5}$$

$$= \frac{2\,0.2 \times 12.5}{4\pi^2 \times 5} \times 10^{-2}$$

$$= 1.28 \times 10^{-2}$$

$$= 1.28\,\%$$ ✓

FINDING RATIO

$$\frac{1.28 \times 10^{-2}}{0.977 \times 10^{-2}} = 1.31$$ ✓

10-4) $R_E = 3.2\ \Omega$; $L_E = 2 \times 10^{-4}$ H

$B = 1\ \frac{\text{WEBER}}{M^2}$; $m = 1.5 \times 10^{-2}$ kg

$R_m = 1\ \frac{\text{Kg}}{\text{SEC}}$; $R_r = 1\ \frac{\text{Kg}}{\text{SEC}}$

$s = 1.5 \times 10^3\ \frac{\text{NEWTONS}}{m}$

$d = 3 \times 10^{-2}$ m ; $N = 80$

$\ell = (2\pi r)N = \pi d N$

$= \pi(3 \times 10^{-2})80$

$= 2.4\pi$ METERS ✓

a) ASSUMING $X_r \approx 0$; $f = 200$ HZ $\Rightarrow \omega = 2\pi f = 1260\ \frac{\text{RAD}}{\text{SEC}}$

i) FIND Z_E (ELECTRICAL IMPEDANCE)

$$\underset{\sim}{Z_E} = R_E + j\omega L_E$$
$$= 3.2 + j(2\pi \cdot 200) \cdot (2 \times 10^{-4})$$
$$= [3.2 + j(0.252)]\ \Omega \quad ✓$$

ii) FIND Z_M (MOTIONAL IMPEDANCE)

$$Z_M = R_M + j X_M$$
$$R_M = \frac{\Phi^2}{Z_m^2}(R_r + R_m)$$
$$X_M = -\frac{\Phi^2}{Z_m^2}\left(X_r + \omega m - \frac{s}{\omega}\right)$$
$$\frac{\Phi}{Z_m} = (B\ell)/(Z_r + Z_c)$$
$$= (B\ell)/\left[(R_r + R_m) + j\left(X_r + \omega m - \frac{s}{\omega}\right)\right]$$
$$= (1 \times 2.4\pi)/\left[(1 + 1) + j\left(0 + 2\pi \times 200 \times .015 - \frac{1}{2\pi}\right.\right.$$
$$= 2.4\pi/\left[2 + j\left(1.5 \times 10^{-3} \times 1.26 \times 10^3 - \frac{15}{4\pi}\right)\right.$$
$$= 2.4\pi/[2 + j(\boxed{1.89} - 1.19) \longrightarrow 18.9$$
$$= 7.54/(2 + j\boxed{0.70})$$
$$\Rightarrow \frac{\Phi^2}{Z_m^2} = (7.54)^2/(2^2 + \boxed{(0.70)^2})$$
$$= 56.8/(4 + 0.49)$$
$$= 56.8/4.49$$
$$= \boxed{1.265}$$

$$R_M = \frac{\phi^2}{Z_m^2}(R_r + R_m)$$

$$= 12.65\,(1 + 1)$$

$$= \boxed{25.3\ \Omega} \qquad .36$$

$$X_M = -\frac{\phi^2}{Z_m^2}\left(X_r + \omega m - \frac{s}{\omega}\right)$$

$$= -12.65\left(0 + 2\pi\cdot 200 \times 0.015 - \frac{1500}{2\pi\cdot 200}\right)$$

$$= -12.65\,(0.70)$$

$$= \boxed{-\ 8.86} \qquad -3.18$$

$$\Rightarrow Z_M = R_M + j\,X_M$$

$$= \boxed{[25.3 - j\ 8.86]\ \Omega} \qquad .36 - 3.18j$$

iii) FIND Z_I (TOTAL ELECTRICAL INPUT IMPEDANCE)

$$Z_I = Z_M + Z_E \ ✓$$

$$= \cancel{[25.3 - j\ 8.86]} + [3.2 + j\,(0.252)]$$

$$= \cancel{[28.5 - j\ 8.61]}\ \Omega$$

$$= 3.56 - 2.93j$$

b) FIND $E_{RMS} \ni e = \sqrt{2}\, E e^{j\omega t}$ GIVES PEAK DISPLACEMENT $d = \sqrt{2}\, d_{RRM}$.

e WILL SET UP CURRENT IN THE COIL:

$$i = e/Z_I$$

WHICH PRODUCES A STEADY STATE VELOCITY:

$$
\begin{aligned}
V &= B\ell i / Z_m \\
&= \frac{\phi}{Z_m} i \\
&= \frac{\phi}{Z_m} \frac{e}{Z_I} \\
&= \frac{\phi}{Z_m} \frac{\sqrt{2}\, E_{RMS}}{Z_I} e^{j\omega t} \\
X &= \int_{t_0}^{t} V\,dt \; ; \; t_0 = 0 \\
&= \frac{\phi}{Z_m} \frac{\sqrt{2}\, E_{RMS}}{Z_I} \int_0^t e^{j\omega t} dt \\
&= \frac{\phi}{j\omega Z_m Z_I} \sqrt{2}\, E_{RMS}\, e^{j\omega t}
\end{aligned}
$$

THE DISPLACEMENT AMPLITUDE IS THUS:

$$
\begin{aligned}
d = \sqrt{2}\, d_{RMS} &= \frac{\phi}{\omega Z_m Z_I} \sqrt{2}\, E_{RMS} \\
d_{RMS} &= \frac{\phi}{\omega Z_m Z_I} E_{RMS} \\
E_{RMS} &= \frac{\omega Z_m Z_I}{\phi} d_{RMS}
\end{aligned}
$$

$$
\begin{aligned}
&= \omega d_{RMS} \left| \frac{Z_m}{\phi} Z_I \right| \\
&= 2\pi \cdot 200 \cdot 10^{-3} \left| \left[\frac{2 + j\,0.70}{7.54} \right] \left[28.5 - j\,8.61 \right] \right| \\
&= \frac{400\pi \times 10^{-3}}{7.54} \left[2^2 + 0.70^2 \right]^{1/2} \left[28.5^2 + 8.61^2 \right]^{1/2} \\
&= \frac{4\pi}{7.54} \times 10^{-1} \left[4 + 0.49 \right]^{1/2} \left[8.13 \times 10^2 + 0.74 \times 10^2 \right. \\
&= 0.167\, (4.49)^{1/2} (8.87)^{1/2} \times 10 \\
&= 1.67 \times 2.12 \times 2.96 \\
&= 10.5 \text{ VOLTS}
\end{aligned}
$$

written the wrong

$\mathcal{E}_{RMS} = 13.7$ volts

c) $$W = \frac{\phi^2 R_r E_{RMS}^2}{Z_m^2 Z_I^2}$$

$$Z_I^2 = 28.5^2 + 8.61^2$$

$$= 887\ \Omega^2$$

$$\Rightarrow W = \left(\frac{\phi^2}{Z_m^2}\right) \frac{R_r E_{RMS}^2}{Z_I^2}$$

$$= (12.65) \frac{(1)(10.5)^2}{8.87 \times 10^2}$$

$$= \frac{1.265\ (1.05)^2}{0.887}$$

$$= 1.57 \text{ WATTS}$$ ✓

d) i). $$R_M = \frac{\phi^2}{R_r + R_M}$$

$$= \frac{(B\ell)^2}{R_r + R_M}$$

$$= \frac{(2.4\pi \times 1)^2}{2}$$

$$= 28.5\,\Omega$$ ✓

ii) $$C_M = \frac{X_r/\omega + m}{\phi^2}$$

$$= \frac{0 + 1.5 \times 10^{-3}}{.57.0 \times 10^{+2}}$$

$$= 2.64 \times 10^{-5}$$

$$= \boxed{26.4}\ \mu F$$

iii) $$L_m = \phi^2/S$$

$$= 57/1.5 \times 10^2$$

$$= 0.379 \text{ HENRIES}$$ ✓

10.8) $a = 0.2\ m$

$m = 4 \times 10^{-2}\ kg$

$R_E = 4\ \Omega$

$L_E = 10^{-4}\ H$

$\phi = 10\ \frac{WEBERS}{m}$

$s = 2 \times 10^{3}\ \frac{NEWTONS}{METER}$

$R_m = 2\ \frac{kg}{SEC}$

a) FIND W IF $E_{RMS}^{2} = 10V$ AND $f = 200\ HZ$

$$W = \frac{\phi^2 R_r E_{RMS}^2}{Z_m^2 Z_I^2}$$

$$Z_m = Z_r + Z_c$$

$$Z_r = R_r + jX_r$$

$$R_r = \rho_0 c \pi a^2 R_1(2ka)$$

$$= (415)\pi(0.2)^2 R_1\left(2 \times \frac{2\pi \times 200}{343} \times 0.2\right)$$

$$= 522\ R_1\left(\frac{80\pi}{343}\right) \quad 160\pi$$

$$= 522\ R_1(0.733)$$

$$= 522(0.0651) \qquad \text{FROM pg 506}$$

$$= \cancel{34.0}\ \frac{kg}{SEC} \quad 12.7$$

$$X_r = \rho_0 c \pi a^2 X_1(2ka)$$

$$= 522\ X_1(\cancel{0.73})$$

$$= 522(0.298) \quad 28.0$$

$$= \cancel{156}\ \frac{kg}{SEC}$$

$$Z_c = R_M + j\left[\omega m - \frac{s}{\omega}\right]$$

$$= 2 + j\left[2\pi \cdot 200 \cdot 4 \times 10^{-2} - \frac{2000}{2 \cdot \pi \cdot 200}\right]$$

$$= 2 + j[50.2 - 1.6]$$

$$= 2 + j[48.6]$$

$$\Rightarrow Z_m = (R_r + R_m) + j\left(X_r + \omega m - \frac{s}{\omega}\right)$$

$$= \left[(34.0 + 2) + j(\underset{28}{156} + 48.6)\right] \quad 14.7 + 76.7j$$

$$= \cancel{34.2 + j204.6}$$

$$Z_m^2 = (34.2)^2 + (205)^2$$

$$= 11.7 \times 10^2 + 4.22 \times 10^4$$

$$= \cancel{4.34 \times 10^4} \quad 6106$$

$$Z_I = Z_E + Z_M$$

$$= (R_E + R_M) + j(\omega L_E + X_M)$$

$$R_M = \frac{\phi^2}{Z_m^2}(R_r + R_m)$$

$$= \frac{10^2}{4.07 \times 10^4}(34.0 + 2)$$

$$= (2.46 \times 10^{-3})(36.0)$$

$$= 88.5 \times 10^{-3}\ \Omega$$

$$X_M = -\frac{\phi^2}{Z_m^2}\left(X_r + \omega m - \frac{S}{\omega}\right)$$

$$= -(2.46 \times 10^{-3})(2.04 \times 10^2)$$

$$= -0.502\ \Omega$$

$$\Rightarrow Z_I = (4 + 88.5 \times 10^{-3}) + j(2\pi 200 \times 10^{-4} - 0.502)$$

$$= 4.09 + j(0.126 - 0.502)$$

$$= 4.09 - j(0.376)$$

$$Z_I^2 = (4.09)^2 + (0.376)^2$$

$$= 16.75 + 0.141$$

$$= \cancel{16.9}\ \Omega$$

19.1

THUS:

$$W = \frac{\phi^2 R_r E_{rms}^2}{Z_m^2 Z_I^2}$$

$$= \frac{(100)(34.0)(100)}{(4.34 \times 10^4)(16.9)}$$

$$= \cancel{0.463}\ \text{WATTS}$$

111

b) $P(r,\theta) = \frac{i\rho c k U_0 a^2}{2\pi} e^{i(\omega t - kr)} \left[\frac{2J_1(ka\sin\theta)}{ka\sin\theta}\right]$

FOR A POINT ON AXIS:

$$P = -\rho c U_0 e^{i\omega t}\left[e^{-ik\sqrt{r^2+a^2}} - e^{ikr}\right]$$

THE PRESSURE AMPLITUDE ON AXIS

$$P = \rho c U_0 \left|e^{-ik\sqrt{r^2+a^2}} - e^{ikr}\right|$$

$$= \rho c U_0 \left[\left(e^{-ik\sqrt{r^2+a^2}} - e^{ikr}\right)\left(e^{ik\sqrt{r^2+a^2}} - e^{-ikr}\right)\right]^{1/2}$$

$$= \rho c U_0 \left|\left[2 - e^{-ik\sqrt{r^2+a^2}} e^{-ikr} - e^{ikr} e^{ik\sqrt{r^2+a^2}}\right]^{1/2}\right|$$

$$= \rho c U_0 \left|\left[2 - e^{ik[r+\sqrt{r^2+a^2}]} - e^{-ik[r+\sqrt{r^2+a^2}]}\right]^{1/2}\right|$$

$$= \rho c U_0 \sqrt{2}\left[1 - \cos^2\left[k(r+\sqrt{r^2+a^2})\right]\right]^{1/2}$$

$$= \sqrt{2}\rho c U_0 \sin k(r+\sqrt{r^2+a^2})$$

U_0 = VELOCITY AMPLITUDE OF "PISTON"

$$V = \frac{\phi}{Z_m} i$$

$$= \frac{e}{Z_E + \phi^2/Z_m}$$

$$= \frac{\sqrt{2}E_{RMS}}{Z_E + \frac{\phi^2}{Z_m^2} Z_m^*} e^{j\omega t} = U_0 e^{j\omega t}$$

$$\Rightarrow U_0 = \sqrt{2}E_{RMS}\left|Z_E + \frac{\phi^2}{Z_m^2} Z_m^*\right|^{-1}$$

$$= \sqrt{2}\,10\left|\left(4 + j\,2\pi\times200\times10^{-4} + \frac{100}{4.34\times10^4}\times(34.2 - j204.6)\right]^{-1}$$

$$= 14.1\left|\left(4 + \frac{10^3(3.42)}{4.34\times10^4}\right) + j\left(4\pi\times10^{-2} - \frac{2.05}{4.34}\right)\right|^{-1}$$

$$= 14.1\left|(4 + 0.0771) + j(12.6\times10^{-2} - 0.473)\right|^{-1}$$

$$= 14.1\left|4.08 + j0.599\right|^{-1}$$

$$= 14.1\left[4.08^2 + 0.599^2\right]^{-1/2}$$

$$= 14.1\left[16.7 + 0.36\right]^{-1/2}$$

$$= 14.1\left[17.1\right]^{-1/2}$$

$$= \frac{14.1}{4.13}$$

$$= \cancel{3.42}$$

$$\dot{U}_0 = \frac{\Phi}{Z_M}\cdot\frac{E_0}{Z_I} = \frac{10}{78.1}\cdot\frac{10}{4.37} = .282\,\frac{m}{se}$$

$$\Rightarrow p = \sqrt{2}\,\rho c U_0 \sin \frac{\omega}{c}\left(r + \sqrt{r^2 + a^2}\right)$$

$$= \sqrt{2}\,(4.15 \times 10^2)(3.43) \sin \frac{2\pi \times 200}{3.43}\left(10 + \sqrt{10^2 + 0.2^2}\right)$$

$$= 20.1 \sin\left[3.66\left(10 + \sqrt{100.04}\right)^{1/2}\right]$$

$$= 20.1 \sin\left[3.66\sqrt{20}\right]$$

$$= 20.1 \sin\left[3.66 \times 4.47\right]$$

$$= 20.1 \sin\left[16.35\right]$$

$$= 20.1 \sin\left[16.35 - 4\pi\right]$$

$$= 20.1 \sin\left[16.35 - 12.6\right]$$

$$= 20.1 \sin\left[3.75\ \text{RAD}\ \frac{180^\circ}{\pi\ \text{RAD}}\right]$$

$$= 20.1 \sin\left[215^\circ\right]$$

$$= -20.1 \sin 35^\circ$$

$$= -20.1\,(0.575)$$

$$P_A = \cancel{11.6}\ \frac{NT}{m}$$

$$P_{ampl} = \left(\frac{\rho c i k \pi a^2 U_0}{2\pi r}\right) \text{wow}$$

distant axial point

$$.86\ \frac{n}{m^2}$$

$$\text{Pressure level} = 20 \log_{10} \frac{P_{ampl}}{\text{Reference Press Ampl}}$$

Power ~~[illegible]~~ output of speaker $= \frac{\Phi^2 R_n E_0}{Z_m^2 Z_I^2}$ but $U_0 = \frac{\Phi E}{Z_m Z_I}$ where U_0 is peak velocity of speaker.

$\therefore$ Power output $= (\text{const}) R_{n2}$ [illegible]

10-11) $a = (2 + \frac{100}{\sqrt{f}})\,\text{cm}$

$= (200 + \frac{10^4}{\sqrt{f}})$ METERS $\qquad 10^2_{Hz} < f \le 10^4\,Hz$

LET $V = V_0 e^{j\omega t} \ni V_0 =$ COMPLEX CONSTANT

FOR A CONSTANT VELOCITY AMPLITUDE, IT MAY BE ASSUMED THE SPEAKER IS FED BY CURRENT $i = I e^{j\omega t}$, AND THE CORRESPONDING OUTPUT POWER IS:

$$W = \tfrac{1}{2} R_r V_0^2$$

WHERE $R_r = \rho_0 c \pi a^2 R_1(2ka)$

$$\Rightarrow W = \frac{V_0^2}{2} \rho_0 c \pi a^2 R_1(2ka)$$

✓

$$= \frac{V_0^2}{2} \rho_0 \left[200 + \frac{10^4}{\sqrt{f}}\right] R_1\left[2k\left(200 + \frac{10^4}{\sqrt{f}}\right)\right]$$

✓

THE EXPANSION OF $R_1(x)$: (pg 179, Eq 7.72)

$$R_1(x) = \frac{x^2}{2 \cdot 4} - \frac{x^4}{2 \cdot 4^2 \cdot 6} + \frac{x^6}{2 \cdot 4^2 \cdot 6^2 \cdot 8} - \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^{2n}}{d_n} \quad ; \quad d_n = 2 \cdot 2(n+1) \prod_{i=1}^{n-1} 4(i+1)^2 \quad ; \; n > 1$$

$$\text{NOW } R_1(2ka) = R_1\left[2k\left(200 + \frac{10^4}{\sqrt{f}}\right)\right]$$

$$= R_1\left[\frac{2}{c} 2\pi f\left(200 + \frac{10^4}{\sqrt{f}}\right)\right]$$

$$= R_1\left[\frac{4\pi}{c}\left(200f + 10^4 f^{3/2}\right)\right]$$

$$= \sum_{n=1}^{\infty} \frac{\left[\frac{4\pi}{c}(200f + 10^4 f^{3/2})\right]^{2n}}{d_n}$$

Method OK. Computer program must be doing something right. Did you check to see if it is computing $R_1(2ka)$ correctly?

$$W = \tfrac{1}{2} R_r V_0^2$$

$$= \tfrac{1}{2} \rho_0 c \pi \left[200 + \frac{10^4}{\sqrt{f}}\right] \sum_{n=1}^{\infty} \frac{\frac{4\pi}{c}(200f + 10^4 f^{3/2})^{2n}}{4(n+1) \prod_{i=1}^{n-1} 4(i+1)^2}$$

NORMALIZING THE OUTPUT POWER WITH RESPECT TO THE SQUARE OF THE VELOCITY AMPLITUDE:

$$\frac{W}{V_0^2} = \tfrac{1}{2} \rho_0 c \pi \left[200 + \frac{10^4}{\sqrt{f}}\right] \sum_{n=1}^{\infty} \frac{\left[\frac{4\pi}{c}(200f + 10^4 f^{3/2})\right]^{2n}}{4(n+1) \prod_{i=1}^{n-1} 4(i+1)^2}$$

THIS CALLS FOR A COMPUTER PROGRAM

$R_n = \rho c \pi a^2 R_1(2ka)$

$= \text{const}\; a^2 R_1(2ka)$

where $a = 2 + \frac{100}{\sqrt{f}}$

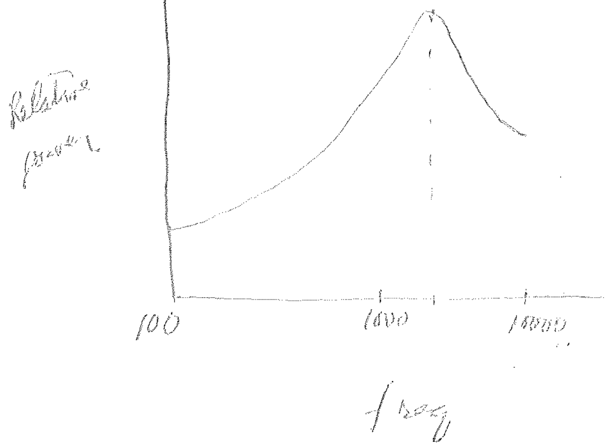
calc $\left(2 + \frac{100}{\sqrt{f}}\right)^2 R_1\left(\frac{2\pi f}{c}\right.$

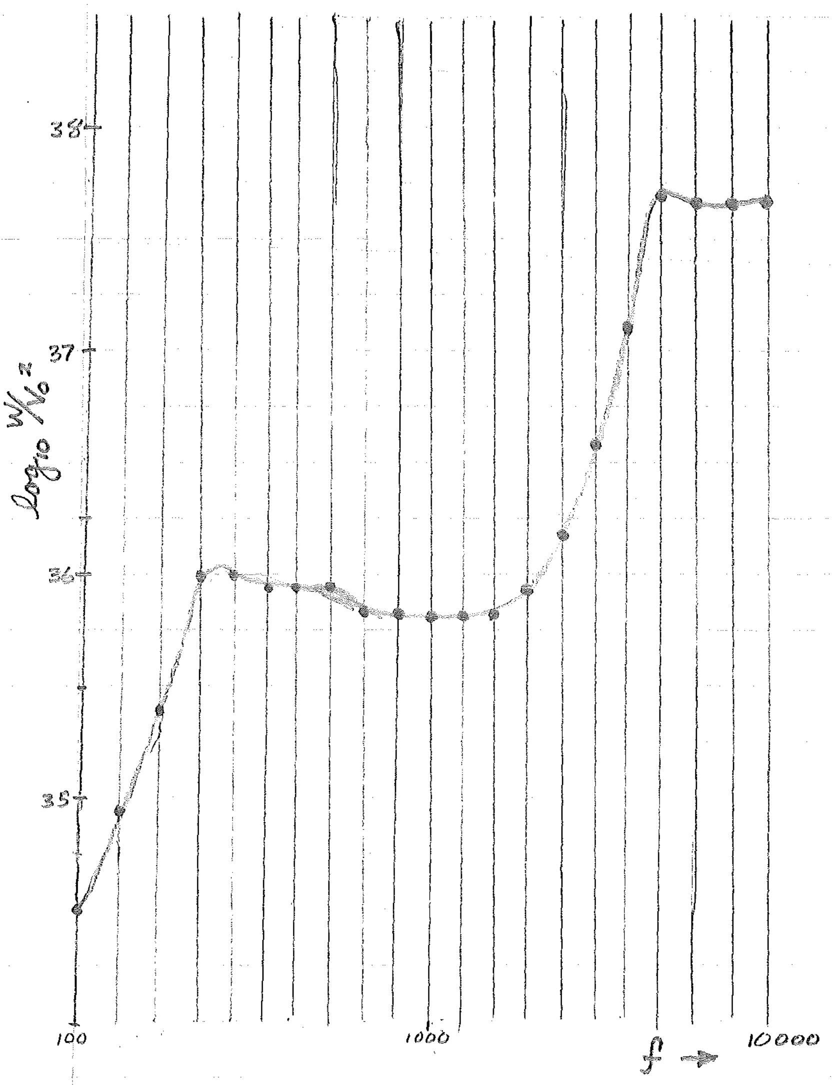
```
          0.158E 03
          0.199E 03
          0.251E 03
          0.316E 03
          0.398E 03
          0.501E 03
          0.630E 03
          0.794E 03
          0.999E 03
          0.125E 04
          0.158E 04
          0.199E 04
          0.251E 04
          0.316E 04
          0.398E 04
          0.501E 04
          0.630E 04
          0.794E 04
          0.999E 04
     S 32 STOP 0000

      EXECUTION TIME 000
```

```
 22222222   777777 77
2222222222  777777
22      22
       22            7
      22
     22
    22              7
   22              77
  22              77
 22               77
2222222222        77
2222222222        77

// JOB T      027
*LIMITS T20.
// FORTRAN MAINLIN  P
*LIST SOURCE PROGR
*IOCS(1403 PRINTER
      READ(2,10)Z,  ,P
      READ(2,11)ND
      WRITE(5,12)
      PI=4.*ATAN(1  )
      FF=ALOG(10.)
      DF=10.**(1./
      COEF=Z*PI/2.
      NF=10.*NDEC+
      DO 50 MF=1,N
      A=P+Q/SQRT(F
      SUM=0.0
      DO 51 N=1,15
      DN=4.*FLOAT(  +1
      DO 52 I=2,N
   52 DN=DN*4.*(FL
   51 SUM=SUM+((4.  PI
      PWR=COEF*A*S
      FLOG=ALOG(F)
      PLOG=ALOG(PW  )/
      WRITE(5,13)F
   50 F=F*DF
      STOP
   10 FORMAT(4F10.
   11 FORMAT(I3,7X
   12 FORMAT(///,1  X,
   13 FORMAT(4(5X,
      END

SEASONS GREETINGS
 IOCS

CORE REQUIREMENTS
 COMMON       0  VA

MERRY CHRISTMAS
// LOAD



           F
     0.100E 03
     0.125E 03       0
```

Should look like this }

← peak around 2000 hertz





38
37
36
35
$\log_{10} W/N_0^2$
100
1000
10000
$f \rightarrow$

10.13) MODELING THE SYSTEM AS AN ACOUSTIC DOUBLET GIVES THE RELATIONSHIP:

$$\frac{P_D}{P_S} = \frac{K^2 \ell^2}{3}$$

(Pg 267)

$$\Rightarrow P_D = \frac{K^2 \ell^2}{3} P_S$$

$$= \frac{\omega^2 \ell^2}{3 C^2} P_s$$

$\ell = 0.4 m$; $f = 100$ HZ ; $P_s = 0.05$ WATTS

$$\Rightarrow P_D = \frac{1}{3}\left(\frac{2\pi f \ell}{c}\right)^2 P_s$$

$$= \frac{1}{3}\left(\frac{2\pi\, 0.4 \times 10^2}{3.43 \times 10^2}\right)^2 (0.05)$$

$$= \frac{(7.33 \times 10^{-1})^2}{3} (5 \times 10^{-2})$$

$= 0.00893$ WATTS

10-16) $m = 10^{-2}$ kg

$s = 10^3 \frac{\text{NEWT}}{m}$

$R_m = 1.5 \frac{kg}{SEC}$

$a = 0.15$ m

$\left.\begin{array}{l} r_{vc} = 1.5 \times 10^{-2}\ m \\ N = 150 \end{array}\right\} \Rightarrow l = 2\pi N r_{vc} = 2\pi(1.5\times10^{2})(1.5\times10^{-2})$

$= 14.15$ m

$B = 0.4 \frac{\text{WEBERS}}{m^2}$

$L_E = 4 \times 10^{-4}$ H

$V = .2 \times 0.5 \times 1 = 0.1\ m^3$

#34 Cu WIRE HAS RESISTANCE OF $\frac{266\,\Omega}{1000\,ft}$

$\Rightarrow R_E = \left(\frac{2.26\,\Omega}{ft}\right)\left(\frac{3.28\,ft}{m}\right)(14.15\,m)$

12.3 Ω

$= \boxed{105\,\Omega}$

a) $S_c = \frac{\rho_0 c^2 (\pi a^2)^2}{V}$

$= (10)(4.15\times10^{2})(3.43\times10^{2})\,\pi^2(1.5\times10^{-1})^4$

$= 4.15 \times 3.43 \times \pi^2 \times 2.25^2 \times 10$

$= 7130 \frac{\text{NEWTONS}}{m}$ ✓

$S_{EFF} = S + S_c$

$= \underline{8.13} \times 10^3$ NEWTON/M

$\omega_0 = \sqrt{S/M}$

$= \left(8.13\times10^{3}/10^{-2}\right)^{1/2}$

$= \sqrt{81.3 \times 10^4}$

$= 902 \frac{RAD}{SEC}$

$f_0 = \frac{\omega_0}{2\pi} = \frac{902}{2\pi} = 144$ HZ if you ignore mass loading

$M_{[illegible]} = .01 + \frac{8}{3}\rho a^3 = .01 + .009 = .019$

$\frac{8}{3}(1.29)(.15)^3$

if you use this instead of $m = .01$ $f_0 \cong 100$ hertz

b) $E_{RMS} = 10$, FIND W

i) AT RESONANCE $(\omega = \omega_0)$

$$W = \frac{\phi^2 R_r E_{RMS}^2}{Z_m^2 Z_I^2}$$

$$\phi = B l$$ 0.8 webers/m²

$$= \boxed{(0.4)}(14.15)$$

$$= \boxed{6.06} \frac{\text{WEBERS}}{m}$$ 11.3

$$\phi^2 = \boxed{36.7} \left(\frac{\text{WEBER}}{m}\right)^2$$

$$R_r = \rho_0 c \pi a^2 R_1\left(\frac{2\omega a}{c}\right)$$

$$= (4.15 \times 10^2)\pi(1.5 \times 10^{-1})^2 R_1\left(\frac{2 \times 9.02 \times 10^2 \times 1.5 \times 1}{3.43 \times 10^2}\right)$$

$$= 4.15 \times \pi \times 1.5^2 R_1\left(\frac{2 \times 9.02 \times 1.5}{3.43} \times 10^{-1}\right)$$

$$= 29.3 \; R_1(0.79)$$ (pg 506)

$$= 29.3 \; (0.075)$$

$$= 2.20 \frac{kg}{sec}$$

$$X_r = \rho_0 c \pi a^2 X_1\left(\frac{2\omega a}{c}\right)$$

$$= 29.3 \; X_1(0.79)$$

$$= 29.3 \; (0.321)$$

$$= 9.42 \frac{kg}{sec}$$

this would be zero if you took into account mass loading

$$Z_m = Z_r + Z_c$$

$$= (R_r + R_m) + j\left(\underbrace{\omega m - \frac{S_{EFF}}{\omega}} + X_r\right)$$

$$= (2.20 + 1.5) + j\,(0 + 9.42$$

$$= 3.7 + j\,9.42$$

$$Z_m^2 = 3.7^2 + 9.42^2$$

$$= 13.7 + 88.7$$

$$= 1.02 \times 10^2$$

$$Z_I = Z_E + Z_M$$

$$= Z_E + \phi^2/Z_m \qquad \frac{\phi^2 Z_m^*}{Z_m^2}$$

$$= (R_E + j\omega L_E) + \frac{36.7}{102}(3.7 - j\,9.42)$$

$$= \boxed{105} + j\,(902)(4 \times 10^{-4}) + \frac{36.7}{102} \times 3.7\Big) + j\left(9.02 \times 4 \times 10^{-2} - \frac{36.7}{102}\,9.4\right.$$

$$= \left(105 + \frac{36.7}{102} \times 3.7\right) + j\,(9.02 \times 4 \times 10^{-2} - \frac{36.7}{102}\,9.4$$

$$= (105 + 1.33) + j\,(36.1 \times 10^{-2} - 3.39)$$

$$= 106 - j\,3.03$$

$Z_I^2 = 1.12 \times 10^4$

$$\Rightarrow W = \frac{\phi^2 R_r E_{RMS}^2}{Z_m^2 Z_I^2}$$

$$= \frac{(36.7)(2.20)10^2}{1.02 \times 10^2 * 1.12 \times 10^4}$$

$$= \frac{(3.67)(2.20)}{(1.02)(1.12)} \times 10^{-4}$$

$$= 7.06 \times 10^{-4} \text{ WATTS}$$ on the small side

Method ok
numbers not correct

Using mass loading & resonant freq of 100 cps

$R_R = 1.0$

$X_R = 6.60$

$Z_m^2 = 6.67$

$Z_I^2 = 3780$

$W = 0.55$ watts

ii) $@\ f = 200\,Hz \Rightarrow \underset{2}{\omega} = 2\pi f = 1.255 \times 10^3$

$$W = \frac{\phi^2 R_r E_{RMS}}{Z_m^2 Z_I^2}$$

$$R_r = \rho_0 c \pi a^2 R_1\left(\frac{2\omega a}{c}\right)$$
$$= 29.3\, R_1\left(\frac{2a}{c}\omega\right)$$
$$= 29.3\, R_1\left(\frac{2 \times 0.15 \times 1.255 \times 10^3}{3.43 \times 10^2}\right)$$
$$= 29.3\, R_1 (1.10)$$
$$= 29.3\,(0.145) \qquad Pg\ 506$$
$$= 4.25\ \frac{kg}{SEC} \quad ✓$$

$$X_r = \rho_0 c \pi a^2 X_1\left(\frac{2\omega a}{c}\right)$$
$$= 29.3\, X_1 (1.10)$$
$$= 29.3\,(0.430) \qquad Pg\ 506$$
$$= 12.6\ \frac{kg}{SEC} \quad ✓$$

$$Z_m = Z_r + Z_c$$
$$= (R_r + R_m) + j\left(\omega m - \frac{S_{EFF}}{\omega} + X_r\right)$$
$$= (4.25 + 1.50) + j\left(1.255 \times 10^3 \times 10^{-2} - \frac{8.13 \times 10^3}{1.255 \times 10^3} + 12.6\right.$$
$$= 5.75 + j\,(12.55 - 6.57 + 12.6)$$
$$= 5.75 + j\,18.6$$

$$Z_m^2 = 5.75^2 + 18.6^2$$
$$= 30.3 + 346$$
$$= 376 \quad ✓$$

$$Z_I = Z_E + Z_m$$
$$= Z_E + \phi^2 / Z_m$$
$$= Z_E + \frac{\phi^2}{Z_m^2} Z_m^*$$
$$= R_E + j\omega L + \frac{\phi^2}{Z_m^2}\left(R_m - j\left(\omega m - \frac{S_{EFF}}{\omega}\right)\right)$$
$$= \boxed{105} + j\,(1.255 \times 10^3)(4 \times 10^{-4}) + \frac{36.7}{376}(5.75 - j\,18.6)$$
$$= 105 + \frac{36.7}{376}\,5.75 + j\left[0.502 - \frac{36.7}{376}\,18.6\right]$$
$$= 105 + 0.558 + j\,[0.502 - 1.81]$$
$$= 106 \boxed{+} j\,1.31$$

$$Z_I^2 = 1.12 \times 10^4$$

$$Z_I^2 = 236$$

$$W = \frac{\phi^2 R_r E_{RMS}^2}{Z_m^2 Z_I^2}$$

$$= \frac{(36.7)(4.25)10^2}{(3.76 \times 10^3)(1.12 \times 10^4)}$$

$$= \frac{(3.67)(4.25)}{(3.76)(1.12)} \times 10^{-4}$$

$$= \boxed{3.71} \times 10^{-4} \text{ WATTS}$$

0.6 watts

iii) $f = 10^3$ Hz $\Rightarrow \omega = 2\pi f = 6.28 \times 10^3$

$W = \frac{\phi^2 R_r E_{RMS}^2}{Z_m^2 Z_I^2}$

$R_r = \rho_0 c \pi a^2 R_1 \left(\frac{2\omega a}{c}\right)$

$= 29.3\ R_1 \left(\boxed{\frac{2 \times 6.28 \times 10^3 \times 0.15}{3.43 \times 10^2}}\right)$

$= 29.3\ R_1 (\boxed{0.548})$

$= (29.3)(0.321 \times 10^{-1}) = \boxed{0.940}$ 33

since for $f = 200$ $R_1 = 1.1$

for $f = 1000$ $R_1 = 5.5$

$X_r = \rho_0 c \pi a^2 X_1 \left(\frac{2\omega a}{c}\right)$

$= 29.3\ X_1 (0.548)$

$= 29.3\ (0.208)$

$= 6.9$ ✓

$Z_m = Z_r + Z_c$

$= (R_r + R_m) + j\left(\omega m - \frac{S_{EFF}}{\omega} + X_r\right)$

$= (\boxed{0.940} + 1.50) + j\left(6.28 \times 10^3 \times 10^{-2} - \frac{8.13 \times 10^3}{6.28 \times 10^3} + 6.9\right)$

$= 2.44 + j\,[62.8 - 1.3 + 6.9]$

$= 2.44 + j\,68.4$

$Z_m^2 = 2.44^2 + 68.4^2$

$= 5.7 + 46.8 \times 10^2$

$= \boxed{46.9 \times 10^2}$ 5850

$Z_I = Z_E + Z_M$

$= Z_E + \frac{\phi^2}{Z_m^2} Z_m^*$

$= (105 + j\,6.28 \times 10^3 \times 4 \times 10^{-4}) + \frac{36.7}{4.69 \times 10^3}[2.44 - j\,68.4]$

$= 105 + \frac{36.7}{4.69 \times 10^3} \times 2.44 + j\left[2.51 - \frac{6.84 \times 3.67 \times 10^3}{4.69 \times 10^3}\right]$

$= 105 + j\,[2.51 - 0.62]$

$= \boxed{105} + j\,1.89$

$Z_I^2 = \boxed{1.10 \times 10^4}$ 172

$W = \frac{\phi^2 R_r E_{RMS}^2}{Z_m^2 Z_I^2}$

$= \frac{(36.7)(0.940)10^2}{(46.9 \times 10^2)(1.10 \times 10^4)}$

$= \frac{(3.67)(9.40)}{(4.69)(1.10)} \times 10^{-5}$

$= \boxed{6.70 \times 10^{-5}}$ WATTS

0.42 watts

10-19) $f = 250\ \text{Hz} \Rightarrow \omega = 2\pi f = 500\pi = 1.57\times10^{3}$

$S_0 = \pi\times(3\times10^{-2})^2 = 9\pi\times10^{-4} = 2.83\times10^{-3}\ \text{m}^2$

$m = 5$

a) $f_c = \dfrac{mc}{4\pi}$

$= \dfrac{(5)(3.43\times10^{3})}{4\pi}$

$= 136.5\ \text{Hz}$ ✓

b) $W = \frac{1}{2} R_r V_{pk}^{2}$ ✓

$U_{pk} = V_{pk} S \Rightarrow V_{pk} = \dfrac{U}{S} \Rightarrow V^2 = \left(\dfrac{U}{S}\right)^2$

SO AT THE THROAT $(x = 0)$

$W = \frac{1}{2}(S_0^2 R_0)\left(\dfrac{U_0^2}{S_0^2}\right)$

$= \frac{1}{2} R_0 U_0^2$

$\Rightarrow U_0 = \sqrt{\dfrac{2W}{R_0}}$

$R_0 = \dfrac{\rho_0 c}{S_0}\sqrt{1 - \dfrac{m^2}{4k^2}}$

$= \dfrac{\rho_0 c}{S_0}\sqrt{1 - \left(\dfrac{mc}{2\omega}\right)^2}$

$= \dfrac{4.15\times10^{2}}{2.83\times10^{-3}}\left[1 - \left\{\dfrac{(5)(3.43\times10^{2})}{2(1.57\times10^{3})}\right\}^2\right]^{1/2}$

$= 1.465\times10^{5}\left[1 - (5.45\times10^{-1})^2\right]^{1/2}$

$= 1.465\times10^{5}\left[1 - 0.297\right]^{1/2}$

$= 1.465\times10^{5}\sqrt{0.703}$

$= (1.465)(0.839)\times10^{5}$

$= 1.23\times10^{5}$

$U_0 = \sqrt{\dfrac{2W}{1.23\times10^{5}}}$

$= \sqrt{1.625\times10^{-5}}$

$= \sqrt{16.25\times10^{-6}}$ ✓

$= 4.06\times10^{-3}\ \dfrac{\text{m}^3}{\text{sec}}$

c) $V = \omega \mathcal{E}_0$

$\Rightarrow V_0 S_{0D} = U_0 = \omega \mathcal{E}_0 S_{0D} \ni S_{0D} =$ DRIVER AREA

OR $\mathcal{E}_0 = \frac{U_0}{\omega S_{0D}}$

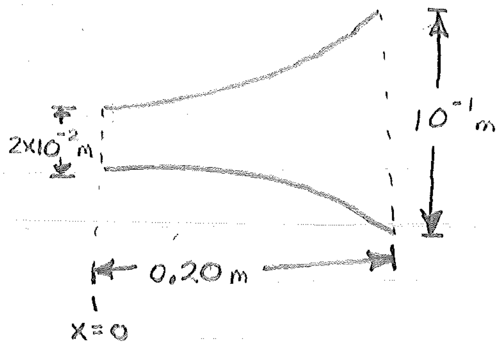
$= \frac{(4.06 \times 10^{-3})}{(1.57 \times 10^{3})(\pi)(5 \times 10^{-2})^2}$

$= \frac{4.06}{(1.57)(\pi)(25)} \times 10^{-3}$

$= 0.330 \times 10^{-3}$

$= 3.30 \times 10^{-4}$ METERS ✓

10-22)



$S_x = S_0 e^{mx}$

$\pi[10^{-1}]^2 = \pi[0.2\times10^{-1}]^2 e^{m(0.20)}$

$e^{m(0.20)} = \frac{10^{-2}}{0.04\times10^{-2}} = 25$

$0.20m = \ln 25$

$m = 5 \ln 25$

$= 5\,[0.219]$

$= 1.095\,/m$

ln 25

FREQUENCY OF PLANEWAVE:

$f = 2\times10^{3} \Rightarrow \omega = 2\pi f = 4\pi\times10^{3} = 1.255\times10^{\overset{4}{3}}\frac{RAD}{SEC}$

PRESSURE:

$n = 74\,db = 20 \log \frac{P}{E} + 74\,db$

$\Rightarrow P = E$ RELATIVE TO 2×10^{-4} MICROBARS/VOLT

OR $P = E\times 2\times10^{-4}$ (MICROBARS)

SO FOR ONE VOLT, THE INCIDENT PRESSURE AMPLITUDE IS 2×10^{-4} MICROBARS

$$\boxed{-\frac{\delta P}{\delta x} = \rho\frac{\delta u}{\delta t} = \rho\frac{\delta^2\xi}{\delta t^2}}$$ OK

There is no reflected wave

incident wave

[illegible] $A = -\frac{P}{\ }$

$$\xi = e^{-\alpha x}e^{j\omega t}\left[Ae^{-j\beta x} + Be^{j\beta x}\right]$$

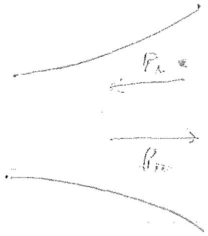
$$u = \frac{\delta\xi}{\delta t} = j\omega e^{-\alpha x}e^{j\omega t}\left[Ae^{-j\beta x} + Be^{j\beta x}\right]$$

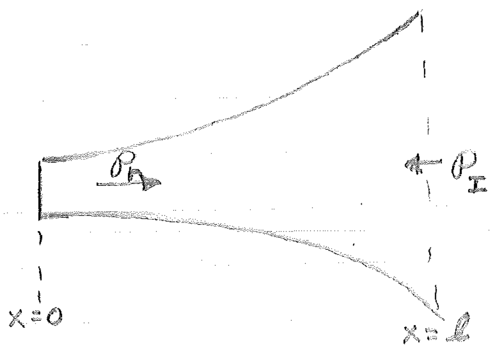
$$\frac{\delta u}{\delta t} = \frac{\delta^2\xi}{\delta t^2} = -\omega^2 e^{-\alpha x}\left[Ae^{-j\beta x} + Be^{j\beta x}\right]e^{j\omega t}$$

$$\frac{\delta P}{\delta x} = -\rho\frac{\delta^2\xi}{\delta t^2} = +\rho\omega^2\left[Ae^{-(j\beta+\alpha)x} + Be^{(j\beta-\alpha)x}\right]e^{j\omega t}$$

$$P = \rho\omega^2\left[\frac{-A}{(j\beta+\alpha)}e^{-(j\beta+\alpha)x} + \frac{B}{(j\beta-\alpha)}e^{(j\beta-\alpha)x}\right]e^{j\omega t}$$

$$\begin{cases} P = \rho\omega^2 e^{-\alpha x}\left[\frac{-A}{(j\beta+\alpha)}e^{-j\beta x} + \frac{B}{(j\beta-\alpha)}e^{j\beta x}\right]e^{j\omega t} \\ u = j\omega\, e^{-\alpha x}\left[Ae^{-j\beta x} + Be^{j\beta x}\right]e^{j\omega t}\end{cases}$$





BOUNDRY CONDITIONS:

$$u\Big|_{x=0} = 0 \qquad \text{(RIGID DIAPHRAM DOES NOT MOVE)}$$

$$u\Big|_{x=0} = 0 = j\omega\left[A + B\right] e^{j\omega t}$$

$$\Rightarrow A = -B \quad \checkmark$$

YIELDING:

$$P = \rho\omega^2 e^{-\alpha x}\left[\frac{A}{(j\beta+\alpha)} e^{-j\beta x} - \frac{A}{(j\beta-\alpha)} e^{j\beta x}\right] e^{j\omega t}$$

$$= \rho\omega^2(\beta^2+\alpha^2)^{-1} e^{-\alpha x}\left[A(j\beta-\alpha) e^{-j\beta x} - A(j\beta+\alpha) e^{j\beta x}\right] e^{j\omega}$$

$$= \frac{\rho\omega^2 e^{-\alpha x}}{(\beta^2+\alpha^2)}\left[jA\beta e^{-j\beta x} - jA\beta e^{j\beta x} - A\alpha e^{-j\beta x} - A\alpha e^{j\beta x}\right] e^{j\omega}$$

$$= \frac{\rho\omega^2 e^{-\alpha x}}{\beta^2+\alpha^2}\left[2\left\{\frac{A\beta e^{j\beta x} - A\beta e^{-j\beta x}}{j2}\right\} - 2\left\{\frac{A\alpha e^{+j\beta x} + A\alpha e^{-j\beta x}}{2}\right\}\right] e^{j}$$

$$= \frac{2\rho\omega^2 e^{-\alpha x}}{\beta^2+\alpha^2}\left[A\beta \sin\beta x - A\alpha\cos\beta x\right] e^{j\omega t}$$

$$= \frac{2A\rho\omega^2 e^{-\alpha x}}{\beta^2+\alpha^2}\left[\beta\sin\beta x - \alpha\cos\beta x\right] e^{j\omega t}$$

$$u = j\omega e^{-\alpha x}\left[A e^{-j\beta x} - A e^{j\beta x}\right] e^{j\omega t}$$

$$= -jA\omega e^{-\alpha x}(j2)\left[\frac{e^{j\beta x} - e^{-j\beta x}}{j2}\right] e^{j\omega t}$$

$$= 2A\omega e^{-\alpha x}\sin\beta x\, e^{j\omega t}$$

THE PRESSURE AMPLITUDE AT $x = l$ IS:

$$P_A = \frac{2A\rho\omega^2 e^{-\alpha l}}{\beta^2+\alpha^2}\left[\beta \sin \beta l - \alpha \cos \beta l\right]$$

This is the pressure ampl at the [illegible] mouth due to both incident and reflected wave. I believe the pressure ampl of the incident wave to be 2×10^{-4} microbars

AND WAS SHOWN TO BE = 2×10^{-4} MICROBARS/VOLT.

SOLVING FOR A:

$$A = \frac{P_A(\beta^2+\alpha^2)}{2\rho\omega^2 e^{-\alpha l}}\left[\beta \sin \beta l - \alpha \cos \beta l\right]^{-1}$$

$$\beta = \sqrt{k^2 - \frac{m^2}{4}}$$

$$= \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m}{2}\right)^2}$$

$$= \left[\left(\frac{1.255\times10^2}{3.43\times10^2}\right)^2 - \left(\frac{1.095}{2}\right)^2\right]^{\frac{1}{2}}$$

$$= \sqrt{13.4 - 0.30}$$

$$= \sqrt{13.1}$$

$$= 3.62/m \quad ; \beta^2 = 13.1$$

$$\alpha = \frac{m}{2}$$

$$= \frac{1.095}{2}$$

$$= 0.547/m \quad ; \alpha^2 = 0.30$$

$$\beta l = (3.62)(0.2) = 7.25 \text{ RAD } \frac{180^\circ}{\pi \text{ RAD}} = 415^\circ$$

$$\sin \beta l = \sin 415^\circ = -\sin 55^\circ = -0.82$$

$$\cos \beta l = \cos 415^\circ = \cos 55^\circ = 0.143$$

$$e^{-\alpha l} = e^{-(0.547)(0.20)}$$

$$= e^{-0.01094}$$

$$= \cosh(0.01094) - \sinh(0.01094)$$

$$= 1.00006 \quad -.01094$$ (CRC TABLES)

$$= 0.989$$

SUBSITUTING:

$$A = \frac{(\cancel{2}\times 10^{-4})[13.1+0.3]}{\cancel{2}\times(1.21)[1.255^2]\times 10^{6}\,(0.989)}\left[3.62\times(-0.82)-(0.547)(0.143)\right]^{-1}$$

$$= \frac{-13.4}{(1.21)(1.255)^2(0.989)}\times 10^{-10}\,(2.97-0.08)^{-1}$$

$$= \frac{-7.12\times 10^{-10}}{2.89}$$

$$= -2.47\times 10^{-10}$$

THE PRESSURE AMPLITUDE AT THE THROAT:

$$|P|_{x=0}| = \frac{-2A\rho\omega^2\alpha}{\alpha^2+\beta^2}$$

$$= \frac{(-2)(-2.47\times10^{-10})(1.21)(1.255)^2\times10^6\ (0.547)}{13.1+0.30}$$

$$= \frac{(2)(2.47)(1.21)(1.255)^2(0.547)}{1.34}\times10^{-5}$$

$$= 3.84\times10^{-5}\ \frac{\text{MICROBARS}}{\text{VOLT}}$$

$$n = 20\log_{10}3.84\times10^{-5} + 47\text{db}$$

$$= -20\log_{10}2.61\times10^{4} + 47\text{db}$$

$$= -20\,[4.42] + 47$$

$$= -41.4\text{db} \text{ RELATIVE TO } 2\times10^{-4}\text{ MICROBARS/VOLT}$$

validity of horn as used to measure the signal at the microphone

Kinsler & Frey

10.22

$f = 2000$ hertz

$w = 2\pi(2000) = 1.25 \times 10^4$

$k = \frac{w}{c} = \frac{2\pi\ 2000}{343} = 36.6$

$k a_L = (36.6)(.1) = 3.66 > 3$

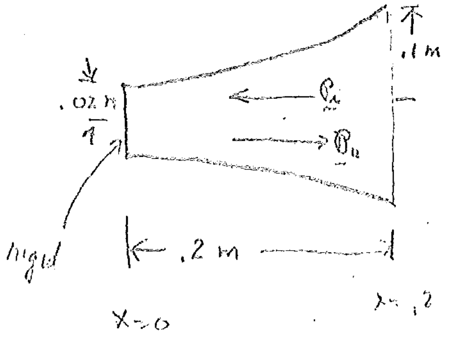
So infinite exponential horn theory applies

$S = S_0 e^{mx}$

$\pi(.1)^2 = \pi(.02)^2 e^{m(.2)}$

$e^{.2m} = 25$

$m = 16.05$



$\alpha = \frac{m}{2} = 8.02$

$\beta = \sqrt{k^2 - \left(\frac{m}{2}\right)^2} = \sqrt{(36.6)^2 - (8.02)^2} = 35.7$

$m - \alpha = \frac{m}{2} = 8.02$

Let $\xi_i = \underset{\sim}{A} e^{-\alpha x} e^{j(\omega t + \beta x)}$ represent particle displacement in the horn due to incident wave

Then from eq 10.36

$$\underset{\sim}{P} = -\frac{\rho c^2}{S_x}\frac{\partial}{\partial x}(S_x \xi) = -\frac{\rho c^2 S_0}{S_0 e^{mx}}\left[m e^{mx} \xi + e^{mx}\frac{\partial \xi}{\partial x}\right]$$

$$= -\rho c^2\left[m\xi + \frac{\partial \xi}{\partial x}\right]$$

$$\therefore \underset{\sim}{P_i} = -\rho c^2\left[m \underset{\sim}{A} e^{-\alpha x} e^{j(\omega t + Bx)} + \underset{\sim}{A}(-\alpha) e^{-\alpha x} e^{j(\omega t + \beta x)} + \underset{\sim}{A} e^{-\alpha x}(j\beta) e^{j(\omega t + Bx)}\right]$$

$$= -\rho c^2 \underset{\sim}{A} e^{-\alpha x}\left[m - \alpha + j\beta\right] e^{j(\omega t + \beta x)}$$

$$= -\rho c^2 |\underset{\sim}{A}| e^{j\phi_1} e^{-\alpha x}\left[(m-\alpha)^2 + \beta^2\right]^{\frac{1}{2}} e^{j\phi_2} e^{j(\omega t + Bx)}$$

$$= -\rho c^2 |\underset{\sim}{A}| e^{-\alpha x}\sqrt{(m-\alpha)^2 + \beta^2}\ e^{j(\omega t + \beta x + \phi_1 + \phi_2)}$$

$$\underset{x=.2}{\underset{\text{ampl}}{P_i}} = \rho c^2 |\underset{\sim}{A}| e^{-.2\alpha}\sqrt{(m-\alpha)^2 + \beta^2}$$

The sound pressure level corresponding to this ampl is 74 db

10.22 cont

Similarly if

$$\underset{\sim}{\xi_r} = \underset{\sim}{B} e^{-\alpha x} e^{j(\omega t-\beta x)}$$

represents the particle displacement due to the reflected wave

$$\underset{\sim}{P_r} = -\rho c^2 \left[m \underset{\sim}{B} e^{-\alpha x} e^{j(\omega t-\beta x)} + \underset{\sim}{B}(-\alpha) e^{-\alpha x} e^{j(\omega t-\beta x)} + \underset{\sim}{B}(-j\beta) e^{-\alpha x} e^{j(\omega t-\beta x)} \right]$$

$$= -\rho c^2 \underset{\sim}{B} e^{-\alpha x} e^{j(\omega t-\beta x)} \left[m - \alpha - j\beta \right]$$

The resultant particle displacement in the horn is

$$\underset{\sim}{\xi} = \underset{\sim}{\xi_i} + \underset{\sim}{\xi_r} = e^{-\alpha x}\left[\underset{\sim}{A} e^{j(\omega t+\beta x)} + \underset{\sim}{B} e^{j(\omega t-\beta x)} \right]$$

Since at $x = 0$ $\quad \underset{\sim}{\xi} = 0 \Rightarrow \underset{\sim}{A} = -\underset{\sim}{B}$. The resultant pressure $\underset{\sim}{P}$ at any point is

$$\underset{\sim}{P} = \underset{\sim}{P_i} + \underset{\sim}{P_r} = -\rho c^2 \underset{\sim}{A} e^{-\alpha x}\left[m-\alpha+j\beta\right] e^{j(\omega t+\beta x)} + \rho c^2 \underset{\sim}{A} e^{-\alpha x}\left[m-\alpha-j\beta\right] e^{j(\omega t-\beta x)}$$

at $x = 0$

$$\underset{\sim}{P}_{x=0} = -\rho c^2 \underset{\sim}{A}\, 2j\beta\, e^{j\omega t} = -\rho c^2 |\underset{\sim}{A}|\, 2\beta\, e^{j(\omega t + \frac{\pi}{2} + \phi_1)}$$

$$P_{\substack{ampl \\ x=0}} = \rho c^2 |\underset{\sim}{A}|\, 2\beta$$

$$20 \log_{10} \frac{P_{\substack{ampl \\ x=0}}}{P_{\substack{i\,ampl \\ x=.2}}} = 20 \log_{10} \frac{\rho c^2 |\underset{\sim}{A}|\, 2\beta}{\rho c^2 |A|\, e^{-.2\alpha} \sqrt{(m-\alpha)^2+\beta^2}} = 20 \log_{10} \frac{2\beta}{e^{-.2\alpha}\sqrt{(m-\alpha)^2+\beta^2}}$$

$$= 20 \log_{10} \frac{(2)(35.7)}{e^{-1.6}\sqrt{(8.02)^2+(35.7)^2}} = 20 \log_{10} \frac{(2)(35.7)}{(.2)(36.6)} = 20 \log_{10} 9.76 = 1$$

$$\text{S.P.L at throat} = 74 + 19.8 = \underline{\underline{93.8 \text{ db}}}$$

Bob Marks
Adv. Acoustics

2 ✓
3 ✓
4 .98
6 .75
9 .85
11 .85
14 .90
6.40
7

11.2) $a = 2 \times 10^{-2}$ m

$d = 2 \times 10^{-5}$ m

$T = 10^{4} \frac{NT}{M}$

a) $E_0 = 200$ V

$$M_c = \frac{E_c}{P}$$
$$= \frac{E_0 a^2}{8 d T}$$
$$= \frac{(2 \times 10^2)(4 \times 10^{-4})}{(8)(2 \times 10^{-5})(10^4)}$$
$$= 5 \times 10^{-2} \frac{\text{VOLTS}}{NT/m^2}$$ ✓

b) $$n = 20 \log_{10}\left[5 \times 10^{-2} \frac{\text{VOLTS}}{NT/M^2}\right)\left(\frac{NT/M^2}{10 \text{ MICROBARS}}\right)\Big]$$
$$= 20 \log_{10} 5 \times 10^{-3} \frac{\text{VOLTS}}{\text{MICROBAR}}$$
$$= -20 \log_{10} 2 \times 10^{2}$$
$$= (-20)(2.3)$$
$$= -46.0 \text{ db Re } \frac{1 \text{ VOLT}}{\text{MICROBAR}}$$ ✓

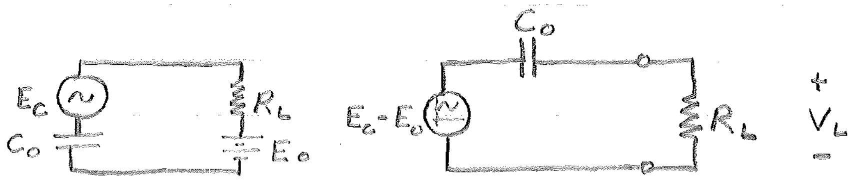
c) $P = 1 \frac{NT}{m^2}$

$$\bar{Y} = \frac{Pa^2}{8T} \sin \omega t$$
$$\bar{Y}_{AMP} = \frac{Pa^2}{8T}$$
$$= \frac{(1)(4 \times 10^{-4})}{8 \times 10^{4}}$$
$$= 5 \times 10^{-9} \text{ m}$$ ✓

d) $R_L = 5 \times 10^{6}\ \Omega$

$f = 10^{2}$ HZ $\Rightarrow \omega = 6.28 \times 10^{2} \frac{RAD}{SEC}$

$P = 10$ MICROBARS $= 1 \frac{NT}{M}$



$$V_L = \frac{R_L}{R_L + \frac{1}{j\omega C}}\left[E_c - E_o\right]$$

THE A.C. COMPONENT OF OUTPUT VOLTAGE IS THEN:

$$V_{L_{AC}} = \frac{R_L E_c}{R_L + \frac{1}{j\omega C_o}}$$

$$V_{L(A.C.)} = \frac{R_L E_C}{R_L + \frac{1}{j\omega C_0}}$$

$$= \frac{R_L M_C P}{R_L - j\frac{d}{\omega \epsilon_0 \pi a^2}}$$

$$= \frac{(5\times10^6)(5\times10^{-2})(1)}{(5\times10^6) - j\frac{2\times10^{-5} \times 10^{12}}{(27.8)(4\times10^{-4})(6.28\times10^2)}}$$

$$= \frac{25\times10^4}{5\times10^6 - j\,2.87\times10^6}$$

$$|V_{L(AC)}| = \frac{25\times10^{-2}}{5.93}$$

✓

$$= 4.22\times10^{-2} \text{ VOLTS} \quad \text{(PEAK)}$$

$$V_{L(D.C.)} = \frac{R_L E_0}{R_L + \frac{1}{j\omega C_0}}$$?

$$|V_{L(D.C.)}| = \frac{5\times2\times10^2}{5.93}$$

?

$$= 169 \text{ VOLTS}$$

13-3) $a = 4 \times 10^{-3}\,m$

$t = 10^{-5}\,m$

$T = 10^{4}\,\frac{NT}{m}$

$d = 10^{-5}\,m$

$E_0 = 150\,V$

a) $f_1 = \frac{2.405}{2\pi a}\sqrt{\frac{T}{\sigma}}$; $J_0(2.405) = 0$; Eq. 4.14 ; Pg 89

$\sigma = \rho_{STEEL} \times t$

$= 7.70 \times 10^{-2}\,\frac{kg}{m^2}$

$f = \frac{2.405}{2\pi(4\times10^{-3})}\sqrt{\frac{10^4}{7.7\times10^{-2}}}$

$= 95.7\sqrt{13\times10^{4}}$

$= 3.45 \times 10^{4}\,HZ$ ✓

b) $f = 10^{4}\,HZ$

$M_c = \frac{E_c}{P}$

$= \frac{E_0 a^2}{8dT}$

$= \frac{(1.5\times10^{2})(16\times10^{-6})}{8\times10^{-5}\times10^{4}}$

$= 3\times10^{-3}\,\frac{VOLTS}{NT/M^2}$

$= 3\times10^{-4}\,\frac{VOLTS}{MICROBAR}$

$n = 20\log_{10} 3\times10^{-4}$

$= -20\log_{10}\frac{1}{3}\times10^{4}$

$= -20\left[\log_{10} 3.33\times10^{3}\right]$

$= -20\left[3.523\right]$

$= -70.4\,db$ ✓

$$Z = \frac{1}{j\omega C_0}$$

$$C_0 = \frac{\epsilon_0 \pi a^2}{d} = \frac{27.8 \times 10^{-12} a^2}{d}$$

$$\omega = 2\pi \times 10^4$$

$$Z = \frac{-j\, d \times 10^{+12} \times 10^{-4}}{27.8 \times 2\pi \times a^2}$$

$$= -j \frac{10^{-5} \times 10^{8} \times 10^{6}}{27.8 \times 2\pi \times 16}$$

$$= \frac{-j\,1}{27.8 \times 2\pi \times 16} \times 10^9$$

$$= \frac{-j}{(.278)(.16) \times 2\pi} \times 10^5$$

$$= -j\,(3.58 \times 10^5)\,\Omega \checkmark$$

c) $ka = \frac{\omega}{c} a = \frac{2\pi \times 10^4}{3.43 \times 10^2} 4 \times 10^{-3} = 0.732$ ✓

ASSUMING NORMAL INCIDENCE $(\theta = 0)$, THE CORRESPONDING P_θ/P READ FROM BALLANTINE'S GRAPH ON Pg 311 $\cong 1.3$

$$db = 20 \log_{10} 1.3$$

$$= 20\,(0.115)$$

$$= 2.3\,db$$

✓

$\Rightarrow$ FREE-FIELD RESPONSE $= -70.4 + 2.3 = -68.1$ db Re $\frac{1 \text{ VOLT}}{\text{MICROBAR}}$

11.4) $M_c = -54db$ Re $\frac{1 \text{ VOLT}}{\text{MICROBAR}}$

$Z_c = 2 \times 10^5 \Omega$ @ 4×10^2 HZ

DECODING:

$$Z_c = \frac{1}{\omega C} = \frac{1}{2\pi f C} \Rightarrow C = \frac{1}{2\pi f Z_c} = \frac{1}{2\pi (4\times10^2)(2\times10^5)} = \frac{1}{50.3 \times 10^7} = \frac{10^{-9}}{0.503} = 1.99 \times 10^{-9} \text{ F}$$

$$-54 db = 20 \log_{10} M_c$$

$$\frac{1}{M_c} = 10^{54/20} = 10^{2.7} = 5.01 \times 10^2$$

$$\Rightarrow M_c = 2 \times 10^{-3} \frac{\text{VOLTS}}{\text{MICROBAR}}$$

a) $P_{DB} = 70$ db Re 2×10^{-4} MICROBAR

$R_L = 5 \times 10^5 \Omega$

$P_{DB} = 20 \log_{10} P_{Re}$ Re 2×10^{-4} MICROBAR

$$7/2 = \log_{10} P \Rightarrow P_{Re} = 10^{3.5} = 3.13 \times 10^3$$

$$\Rightarrow P = (3.13 \times 10^3)(2 \times 10^{-4}) = 6.26 \times 10^{-1}$$

$= 0.626$ MICROBARS

$$M_c = \frac{E_c}{P} \Rightarrow E_c = M_c P = (2\times10^{-3})(0.626) = 1.25 \times 10^{-3} \text{ VOLTS}$$ RMS ✓

$$V_L = \frac{R_L}{R_L - jX_c} E_c$$

this is RMS

$$= \frac{5}{5 - j2}\left[1.25 \times 10^{-3} \text{ VOLTS}\right]$$

$$V_{L_{RMS}} = \frac{5 \times 1.25 \times 10^{-3}}{\sqrt{29}\ \cancel{\sqrt{2}}} = \left(8.21 \times 10^{-4} \text{ V} \times \sqrt{2}\right) = 1.16 \times 10^{-3} \text{ volts}$$

b) $$W = \frac{V_{L_{RMS}}^2}{R_L} = \frac{(2)\,67.5 \times 10^{-8}}{5 \times 10^5} = 2(1.35) \times 10^{-12} \text{ WATTS}$$

2.70×10^{-12}

c) $S = 4 \times 10^{-4}\ m^2$

FOR A NORMAL PLANE WAVE, THE INTENSITY IS:

$$I = \frac{p^2}{2\rho c}$$

$$= \frac{\left[(6.26 \times 10^{-1}\ \text{MICROBARS}) \times \frac{1\ \text{NT}/\text{M}^2}{10\ \text{MICROBARS}}\right]^2}{2 \times 4.15 \times 10^2}$$

$$= 4.21 \times 10^{-6}\ \frac{\text{WATTS}}{m^2}$$

$$W = PS$$

$$= (4 \times 10^{-4})(4.21 \times 10^{-6})$$

$$= 16.84 \times 10^{-10}$$

$$= 1.68 \times 10^{-9}\ \text{WATTS}$$

$$\frac{P_{OUT}}{P_{IN}} = \frac{1.35 \times 10^{-12}}{1.68 \times 10^{-9}} = 8.04 \times 10^{-4}$$ ✓

11.6) $m = 3 \times 10^{-3}$ kg

$a = 5 \times 10^{-2}$ m

$R_m = 10 \frac{kg}{m}$

$s = 5 \times 10^{4} \frac{NT}{m}$

$B = 0.75 \frac{WEBERS}{m^2}$

$l = 10$ m

$R_E = 1\ \Omega$

$L_E = 10^{-5}$ H

$f = 1.1 \times 10^3$ HZ $\Rightarrow \omega = 2\pi f = 6.91 \times 10^3 \frac{RAD}{SEC}$

a) $e = B l V$; $V = f/Z_m$

$\Rightarrow e = \frac{B l f}{Z_m}$

$\frac{e}{f} = \frac{B l}{Z_m}$

$M_m = \frac{e}{f/S} = \frac{B l S}{Z_m}$

MODELING AS A SIMPLE OSCILLATOR:

$Z_m = R_m + j\left(\omega m - \frac{s}{\omega}\right)$

$= 10 + j\left[(6.91 \times 10^3)(3 \times 10^{-3}) - \frac{(5 \times 10^4)}{(6.91 \times 10^3)}\right]$

$= 10 + j[20.8 - 7.2]$

$= 10 + j13.6$

$|Z_m| = 16.9 \frac{kg}{m}$

$S = \pi a^2$

$= \pi \times 25 \times 10^{-4}$

$= 7.86 \times 10^{-3}$ m^2

$M_m = \frac{B l S}{Z_m} = \frac{(0.75)(10)(7.86 \times 10^{-3})}{16.9}$

$= 3.48 \times 10^{-3} \frac{VOLTS}{NT/M^2}$

$n_m = 20 \log_{10} .348 \times 10^{-2} \frac{VOLTS}{NT/M^2} \times \frac{NT/M^2}{10 \text{ MICROBARS}}$

$= -20 \log_{10} 2.87 \times 10^3$ Re $\frac{1 V}{MICROBAR}$

$= (-20)(3.46)$

$= -69.2$ db Re $\frac{1 V}{MICROBAR}$

✓

assuming no loading

b) DETERMINATION OF PROPER LOAD RESISTOR

Z_I = INTERNAL IMPEDANCE

$$Z_I = Z_E + \frac{\phi^2}{Z_m} \quad \ni \phi = B\ell$$

$$= Z_E + \frac{B^2\ell^2}{Z_m}$$

$$= Z_E + \left(\frac{B\ell}{Z_m}\right)^2 Z_m$$

??

$$Re[Z_I] = R_I = R_E + \left(\frac{B\ell}{Z_m}\right)^2 R_m$$

$$= 1 + \left[\frac{(0.75)(10)}{16.9}\right]^2 10$$

$$= 1 + [0.444]^2 10$$

$$= 1 + 1.97$$

$$= 2.97\ \Omega$$

?

SO LET $R_L = 2.97\ \Omega$

$$W = \frac{E^2}{4R_L}$$

$$\frac{W}{P^2} = \frac{1}{4R_L}\left(\frac{E}{P}\right)^2$$

$$n = 10 \log_{10} \frac{1}{4R_L}\left(\frac{E}{P}\right)^2$$

$$= 10 \log_{10} \frac{1}{4(2.97)}\left[3.48 \times 10^{-3}\right]^2 \frac{\text{WATTS}}{[\text{NT}/m^2]^2}\left[\frac{\text{NT}/m^2}{10\text{MICRO}}\right]^2$$

$$= 10 \log_{10} \frac{(3.48)^2 \times 10^{-9}}{4 \times 2.97} \frac{\text{WATTS}}{(\text{MICROBAR})^2}$$

$$= 10 \log_{10} \frac{(3.48)^2 \times 10^{-4}}{4 \times 2.97} \frac{10^{-3}\ \text{WATTS}}{[10\ \text{MICROBARS}]^2}$$

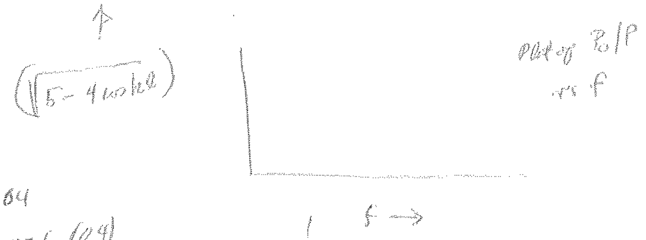
$$= 10 \log_{10} 1.02 \times 10^{-4} \frac{10^{-3}\ \text{WATTS}}{[10\ \text{MICROBARS}]^2}$$

$$= -10 \log_{10} 9.8 \times 10^3$$

$$= (-10)(3.99)$$

$$= -39.9\ \text{db}\ Re\ 10^{-3}\ \text{WATTS @ 10 MICROBARS}$$

?



$l \approx .04$

$kl = \frac{2\pi f(.04)}{c}$

11:9) THE PRESSURE AMPLITUDE ON THE FRONT SURFACE OF THE MOVING ELEMENT OF A VELOCITY RIBBON MICROPHONE MOUNTED IN A CIRCULAR BAFFLE OF RADIUS l IS GIVEN BY eq. 11.37 AS:

$$P_0 = P\sqrt{5-4\cos kl} \quad = \frac{P_0}{P} = \sqrt{5-4\cos kl}$$ ∴ Plot

WHERE P IS THE PRESSURE AMPLITUDE OF THE WAVE INCIDENT AT AN ANGLE θ:

$$\underset{\sim}{p} = Pe^{j(\omega t - kx\cos\theta - ky\sin\theta)} \simeq Pe^{j(\omega t - kx\cos\theta)}$$

FOR NORMAL INCIDENCE $(\theta = 0)$:

$$\underset{\sim}{p} = Pe^{j(\omega t - kx)} \qquad \Rightarrow k = \frac{\omega}{c} = \frac{2\pi f}{c}$$

THE TOTAL FORCE ON THE RIBBON OF SURFACE AREA S IS THEN:

$$\underset{\sim}{F} = [\underset{\sim}{p}_{FRONT} - \underset{\sim}{p}_{BACK}]S$$

THE PRESSURE AMPLITUDE ON THE RIBBON'S BACK IS EQUIVALENT TO THAT OF THE INCIDENT WAVE. THUS:

$$\underset{\sim}{F} = [P_0 e^{j(\omega t - k(0)\cos\theta)} - Pe^{j(\omega t - kl\cos\theta)}]S$$

$$= [P_0 - Pe^{-jkl\cos\theta}]Se^{j\omega t}$$

$$= [\sqrt{5-4\cos kl} - e^{-jkl\cos\theta}]PSe^{j\omega t}$$

$$= [\sqrt{5-4\cos kl} - \cos(kl\cos\theta) + j\sin(kl\cos\theta)]PSe^{j\omega t}$$

FOR NORMAL INCIDENCE:

$$\underset{\sim}{F} = [\sqrt{5-4\cos kl} - \cos kl + j\sin kl]PSe^{j\omega t}$$

$$= \sqrt{(\sqrt{5-4\cos kl} - \cos kl)^2 + \sin^2 kl} \quad PSe^{j(\omega t + \phi)}$$

angle of force vs f

$$\frac{|\underset{\sim}{F}|}{PS} = \sqrt{(\sqrt{5-4\cos kl} - \cos kl)^2 + \sin^2 kl}$$ Plot This vs. f.

THE SPATIAL FORCE PHASE IS THEN:

$$\angle \underset{\sim}{F} = \text{atan} \frac{\text{Im } \underset{\sim}{F}_{SPACE}}{\text{Re } \underset{\sim}{F}_{SPACE}}$$

$$= \text{atan}\left[\frac{\sin(kl\cos\theta)}{\sqrt{5-4\cos kl} - \cos(kl\cos\theta)}\right]$$

FOR NORMAL INCIDENCE:

$$\angle \underset{\sim}{F} = \text{atan}\left[\frac{\sin kl}{\sqrt{5-4\cos kl} - \cos kl}\right]$$

ASSUMING THE SYSTEM IS MASS CONTROLLED, THE VELOCITY AMPLITUDE IS: $(\theta = 0)$

$$V = \frac{f}{j\omega m}$$

$$= \frac{ps\,e^{j\omega t}}{j\omega m}\left[\sqrt{5-4\cos kl} - e^{jkl}\right]$$

THE VELOCITY AMPLITUDE IS THEN:

$$V_{AMP} = \frac{ps}{\omega m}\left|\sqrt{5-4\cos kl} - e^{jkl}\right|$$

$$= \frac{ps}{\omega m}\left|\sqrt{5-4\cos kl} - \cos kl + j\sin kl\right|$$

$$= \frac{ps}{\omega m}\left[\left\{\sqrt{5-4\cos kl} - \cos kl\right\}^2 + \sin^2 kl\right]^{1/2}$$

$$= \frac{ps}{\omega m}\left[(5-4\cos kl) - 2\sqrt{5-4\cos kl}\cos kl + 1\right]^{1/2}$$

$$= \frac{ps}{\omega m}\left[4 - 4\cos kl - 2\sqrt{5-4\cos kl}\cos kl\right]^{1/2}$$

$$= \frac{\sqrt{2}\,ps}{\omega m}\left[2 - \cos kl\left\{2 - \sqrt{5-4\cos kl}\right\}\right]^{1/2}$$

$$= \frac{\sqrt{2}\,ps}{2\pi f m}\left[2 - \cos\frac{2\pi f l}{c}\left\{2 - \sqrt{5-4\cos\frac{2\pi f l}{c}}\right]^{1/2}$$

plot this over f [illegible] vs f

THE VOLTAGE GENERATED IS:

$$\underset{\sim}{e} = B \ell_c \underset{\sim}{V}$$

THUS:

$$|\underset{\sim}{e}| = B \ell_c |\underset{\sim}{V}|$$

$$E = B \ell_c V_{AMP}$$

$$= \frac{\sqrt{2}\, P S B \ell_c}{\omega m}\left[2 - \cos k\ell \left\{2 - \sqrt{5 - 4\cos k\ell}\right\}\right]^{1/2}$$

AND:

$$M_V = \frac{E}{P} = \frac{\sqrt{2}\, S B \ell_c}{\omega m}\left[2 - \cos k\ell \left\{2 - \sqrt{5 - 4\cos k\ell}\right\}\right]^{1/2}$$

$$= \frac{\sqrt{2}\, S B \ell_c}{2\pi f m}\left[2 - \cos \frac{2\pi f \ell}{c} \left\{2 - \sqrt{5 - 4\cos \frac{2\pi f \ell}{c}}\right\}\right]^{1/2}$$

PROBLEM DONE THIS WAY DUE TO LACK OF PROBLEM COMPREHENSION

11-11) $P = 2P \cos \omega t \sin kx$

a) FOR CYLINDER LENGTH $l \ll 1/K$

THE NET FORCE ACTING TO DISPLACE THE CYLYNDER IS GIVEN BY:

$$f = -\frac{\delta P}{\delta x} lS$$
$$= -lS \frac{\delta}{\delta x}\left[2P \cos \omega t \sin kx\right]$$
$$= -2lkPS \cos \omega t \cos kx$$

b) $P_{\text{ANTINODES}}$ OCCUR WHEN $\sin kx = 1$ $(P_{\text{AMPLITUD}} = 2P)$

$$\Rightarrow x = \frac{(2n+1)\pi}{2K} \quad ; \quad n = 0, 1, 2, \ldots \;;$$

SO, AT PRESSURE ANTINODES:

$$f = -2lkPS \cos \omega t \cos k\left[\frac{(2n+1)\pi}{2k}\right]$$
$$= -2lkPS \cos \omega t \cos \frac{(2n+1)\pi}{2}$$
$$= 0$$

THE CORRESPONDING VELOCITY EXPRESSION IS GIVEN BY:

particle velocity antinodes

$$V = f/Z_m = \frac{-2lkPS}{Z_m} \cos \omega t \sin kx$$

AT THE PRESSURE ANTINODES:

$$V = \frac{-2lkPS}{Z_m} \cos \omega t \sin k\left[\frac{(2n+1)\pi}{2k}\right]$$
$$= \frac{-2lkPS}{Z_m} \cos \omega t$$

$V_{\text{AMPLITUDE}} = \frac{2lkPS}{|Z_m|}$ IS MAXIMUM $\Rightarrow$ WE ARE AT THE ANTINODES OF VELOCITY.

11.14) $M_A = 5 M_B$ ✓

$d = 1.5\ m$

$E_A' = 10^{-3}\ V$ ✓

$I_B = 1$ AMP ✓

$f = 500$ HZ

a) $$M_A^2 = \frac{2 d \lambda E_A'}{\rho_0 c I_B} \times \frac{E_A}{E_B}$$

$$= \frac{2 d \lambda E_A'}{\rho_0 c I_B} \times \frac{M_A}{M_B}$$

$$= \frac{10 d \lambda E_A'}{\rho_0 c I_B}$$

$$= \frac{10 d (c/f) E_A'}{\rho_0 c I_B}$$

$$= \frac{10 d E_A'}{\rho_0 I_B f}$$

$$= \frac{10 \times 1.5 \times 10^{-3}}{1.21 \times 1\ (500)}$$

$$= \frac{1.24 \times 10^{-2}}{500}$$

$$\Rightarrow M_A = \overset{4.98 \times 10^{-3}}{\cancel{0.111}} \frac{\text{VOLT}}{\text{NT}/\text{M}^2} = \overset{4.98 \times 10^{-4}}{\cancel{1.11}} \frac{\text{VOLT}}{\text{MICROBAR}}$$

b) $$P_B = E_A' / M_A$$

$$= 10^{-3} / \cancel{1.11}\ 4.98 \times 10^{-4}$$

$$= \cancel{9 \times 10^{-2}} \text{ MICROBARS}$$

.02

12.1) X -CUT

$l_x = 5 \times 10^{-3} m$

$l_Y = 3 \times 10^{-2} m$

$l_Z = 10^{-2} m$

a) $E_X = 10^2 V$

$\frac{1}{3} l_Y \geq l_x, l_z$

$$\frac{\delta \eta}{\delta Y} = -S_{22}\left(\frac{F_Y}{S_Y}\right) + \frac{d_{12} E_x}{l_x}$$

$$= \frac{1}{Y} \frac{(0)}{l_x \, l_z} + \frac{(2.3 \times 10^{-12})(10^2)}{5 \times 10^{-3}}$$

$$= 0.46 \times 10^{-7}$$

$$= 4.6 \times 10^{-8}$$

b) $$\frac{\delta \eta}{\delta Y} = -S_{22}\left(\frac{F_Y}{S_Y}\right) + \frac{d_{12} E_x}{l_x}$$

$$\Rightarrow \frac{F_Y}{S_Y} = \frac{-1}{S_{22}}\left(\frac{\delta \eta}{\delta Y} - \frac{d_{12} E_x}{l_x}\right)$$

FOR RESTRICTED LONGITUDINAL EXPANSION; $\frac{\delta \eta}{\delta Y} = 0$

$$\Rightarrow \frac{F_Y}{S_Y} = \frac{1}{S_{22}} \frac{d_{12} E_x}{l_x}$$

$$= \frac{Y d_{12} E_x}{l_x}$$

$$= \frac{(7.9 \times 10^{10})(2.3 \times 10^{-12})(10^2)}{5 \times 10^{-3}}$$

$$= 3.63 \times 10^3 \frac{NT}{m^2}$$

BOB MARKS

1	.70
3	1.00
8	0.90
13	1.00
16	
19	
23	1.0
	4.6
	7

12.1) X-CUT

$l_x = 5 \times 10^{-3}$

$l_y = 3 \times 10^{-2}$

$l_z = 10^{-2}$

a) $E_x = 100$ V

$$\frac{\delta \eta}{\delta Y} = -s_{22}\left(\frac{F_Y}{S_Y}\right) + \frac{d_{12} E_x}{l_x}$$

FOR UNCONSTRAINED CRYSTAL, $F_Y = 0$

$$\frac{\delta \eta}{\delta Y} = \frac{d_{12} E_x}{l}$$

$$= \frac{(2.3 \times 10^{-12})(10^2)}{5 \times 10^{-3}}$$

$$= 0.46 \times 10^{-7}$$

$$= 4.6 \times 10^{-8}$$

b) $$\frac{\delta \eta}{\delta Y} = -s_{22}\left(\frac{F_Y}{S_Y}\right) + \frac{d_{12} E_x}{l_x}$$

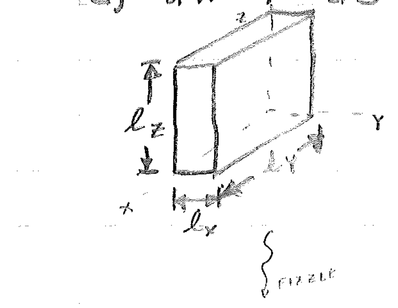
FOR CONSTRAINED CRYSTAL, $\frac{\delta \eta}{\delta Y} = 0$

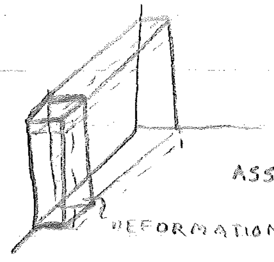
$$\frac{F_Y}{S_Y} = \frac{d_{12} E_x}{S_{22} l_x}$$

$$= \frac{4.6 \times 10^{-8}}{1.27 \times 10^{-11}}$$

$$= 3.62 \times 10^{3} \ \frac{NT}{m^2}$$

c) $dW = F \cdot dS$



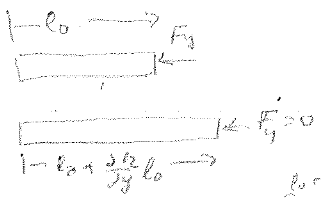


$$\frac{\delta \eta}{\delta Y} = \frac{d_{12} E_x}{l_x}$$

E FIELD $= \frac{E_x}{l_x}$

ASSUME THERE ARE NO EXTERNAL LOSSES IN THE CRYSTAL

Calc. several ways. ①



Energy of deformation $= \left(\frac{1}{2} q V\right)_{\text{unclamped}} - \left(\frac{1}{2} q V\right)_{\text{clamped}}$

$$= \frac{1}{2}\left(\frac{\epsilon_x \epsilon_0 E_x}{l_x} l_y l_z\right) E_x - \frac{1}{2}\left(\frac{\epsilon_x' \epsilon_0 E_x}{l_x} l_y l_z\right) E_x$$

Energy of deformation $= \int_{l_0}^{l_0 + \frac{\partial \eta}{\partial y} l_0} F_y \, dy \cong (F_{y\,av}) \frac{\partial \eta}{\partial y} l_0$

$$\approx \frac{F_y + 0}{2} \frac{\partial \eta}{\partial y} l_0$$

② When voltage is applied and crystal is clamped Force appears & is given by F_y where $F_y = \frac{d_{12} E_x}{l_x} \frac{S_y}{s_{22}}$

③ Now let clamp to be slowly opened. The crystal expands to a length $l_0 + \frac{\partial \eta}{\partial y} l_0$ and the force F_y goes to zero, 12.3)A) $\eta = (A e^{jkY} + B e^{jkY}) e^{j\omega t}$

$\eta|_{Y=0} = 0$

$\eta|_{Y=0} = 0 = (A + B) e^{j\omega t}$

$\Rightarrow A = -B$

$\therefore \eta = A(e^{-jkY} - e^{jkY}) e^{j\omega t}$

$= -A(e^{jkY} - e^{-jkY}) e^{j\omega t}$

$= -j2A \sin kY e^{j\omega t}$

$= A' \sin kY e^{j\omega t}$

$\left.\frac{\delta \eta}{\delta Y}\right|_{Y=l} = 0 = A'k \cos kl_Y e^{j\omega t}$

$\Rightarrow kl = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$= \frac{(2n+1)\pi}{2}$; $n = 0, 1, 2, \ldots$

FOR FUNDEMENTAL FREQ. :

$kl = \frac{\omega_o}{c_Y} l = \frac{\pi}{2}$

$2\pi f_o = \frac{\pi c_Y}{2 l}$

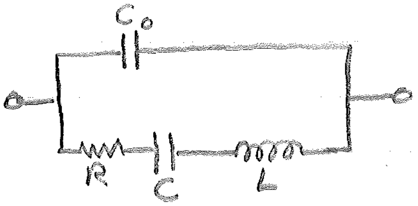
$f_o = \frac{c_Y}{4 l_Y}$

$= \frac{5.45 \times 10^3}{4 \times 3 \times 10^{-2}}$

$= \frac{5.45 \times 10^3}{1.2 \times 10^{-1}}$

$= 4.54 \times 10^4$ HZ ✓

b)



$$Q = 2 \times 10^4$$

$$\phi^2 = \left(\frac{d_{12}\, l_z}{S_{22}}\right)^2$$
$$= \left[\frac{(2.3 \times 10^{-12})(10^{-2})}{1.27 \times 10^{-11}}\right]^2$$
$$= 3.28 \times 10^{-6}$$

$$L = \frac{\rho\, l_Y\, S_Y}{2\phi^2}$$
$$= \frac{(2.65 \times 10^3)(15 \times 10^{-7})}{6.56 \times 10^{-6}}$$
$$= 607\ H$$

$$C = \frac{8\,\phi^2\, S_{22}\, l_Y}{\pi^2\, S_Y}$$
$$= \frac{8 \times 3.28 \times 10^{-6} \times 1.27 \times 10^{-11} \times 3 \times 10^{-2}}{\pi^2 \times 5 \times 10^{-5}}$$
$$= 2 \times 10^{-14}\ F$$

$$Q = \frac{\omega_1 L}{R} \Rightarrow R = \frac{\omega_1 L}{Q}$$
$$= \frac{2\pi \times 4.54 \times 10^4 \times 6.07 \times 10^2}{2 \times 10^4}$$
$$= 8.66 \times 10^3\ \Omega$$

$$R = \frac{\rho_0 C_0 S_Y}{\phi^2} \Rightarrow \rho_0 C_0 = \frac{R\phi^2}{S_Y}$$
$$= \frac{(8.66 \times 10^3)(3.28 \times 10^{-6})}{5 \times 10^{-5}}$$
$$= 5.68 \times 10^2$$

$$C_0 = \frac{\epsilon_x' \epsilon_0\, l_Y\, l_Z}{l_X}$$
$$= \frac{(4.45)(8.85 \times 10^{-12})(3 \times 10^{-4})}{5 \times 10^{-3}}$$
$$= 2.36 \times 10^{-12}\ F$$

$5\ \frac{\text{watts}}{m^2}\ \frac{m^2\ 10000}{1}$

100

12·13) a) $\ell_x = 3 \times 10^{-3}$

$$f_1 = \frac{c_x}{2\ell_x}$$

$$= \frac{5.75 \times 10^3}{6 \times 10^{-3}}$$

$$= 9.6 \times 10^5 \text{ HZ}$$

b) $W = \frac{\phi^2 E_x^2}{\rho_0 c_0} \frac{1}{S_x}$

$$= \left(\frac{2 e_{11} S_x}{\ell_x}\right)^2 \frac{E_x^2}{\rho_0 c_0 S_x}$$

$$= \left(\frac{2 e_{11} S_x}{\ell_x}\right)^2 \frac{S_x}{\rho_0 c_0}$$

$$\frac{W}{S_x} = \left(\frac{2 e_{11} E_x}{\ell_x}\right)^2 \frac{1}{\rho_0 c_0}$$

$$E_x^2 = \left(\frac{W}{S_x}\right) \rho_0 c_0 \left(\frac{\ell_x}{2 e_{11}}\right)$$

$$= (5 \times 10^4)(1.48 \times 10^6)\left[\frac{3 \times 10^{-3}}{0.34}\right]^2$$

$$= 7.41 \times 10^{10} (77.4 \times 10^{-6})$$

$$= 574 \times 10^4$$

$\Rightarrow E_x = 2400$ VOLTS ✓

→ ~~c) $W_{H_2O} = \frac{\phi^2 E_x^2}{(\rho_0 c_0)_{H_2O}} \frac{1}{S_x}$; $W_{AIR} = \frac{\phi^2 E_x^2}{(\rho_0 c_0)_{AIR}} \frac{1}{S_x}$~~

~~$W_{H_2O} (\rho_0 c_0)_{H_2O} = W_{AIR} (\rho_0 c_0)_{AIR}$~~

~~$\therefore \frac{W_{AIR}}{S_x} = \frac{W_{H_2O}}{S_x} \frac{(\rho_0 c_0)_{H_2O}}{(\rho_0 c_0)_{AIR}}$~~

~~$= (5) \frac{(14.8 \times 10^5)}{(4.15 \times 10^2)}$~~

~~$= 17.8 \times 10^3$~~

~~$= 1.78 \times 10^4 \frac{\text{WATTS}}{\text{CM}^2}$~~

C) AT RESONANCE

$$X_{AMP} \propto \frac{E}{\rho_0 c_0} \qquad (\text{cf Eq. 12.8}) \qquad (\propto = \text{PROPORTIONAL TO})$$

THEN IN WATER:

$$d_{H_2O} \propto \left(\frac{E}{\rho_0 c_0}\right)_{H_2O}$$

SO FOR EQUAL AMPLITUDE DISPLACEMENTS IN AIR & WATER

$$\left(\frac{E}{\rho c}\right)_{H_2O} = \left(\frac{E}{\rho c}\right)_{AIR}$$

$$E_{AIR} = (\rho c)_{AIR} \left(\frac{E}{\rho c}\right)_{H_2O}$$

THE POWER OUT IN AIR IS

$$W = \frac{\phi^2 E_{AIR}^2}{(\rho c)_{AIR} S_x}$$

AND THE INTENSITY:

$$I = W/S_x = \left[\frac{2 e_{11} E_{AIR}}{l_x}\right]^2 \frac{1}{(\rho c)_{AIR}}$$

$$= \left[\frac{2 e_{11} (\rho c)_{AIR} E_{H_2O}}{l_x (\rho c_{H_2O})}\right]^2 \frac{1}{(\rho c)_{AIR}}$$

$$= \left[\frac{2 e_{11} E_{H_2O}}{(\rho c)_{H_2O} l_x}\right]^2 \rho c_{AIR}$$

$$= \left[\frac{2 \times 0.17 \times 2.4 \times 10^{3}}{(1.48 \times 10^{6})(3 \times 10^{-3})}\right]^2 \times 4.15 \times 10^2$$

$$= (1.835 \times 10^{-1})^2 \times 4.15 \times 10^2$$

$$= 14.0 \ \frac{\text{WATTS}}{m^2} \times \frac{1 \ m^2}{10^4 cm^2}$$

$$= 1.4 \times 10^{-3} \ \frac{\text{WATTS}}{cm^2}$$ ✓

12.8) $\ell_y = 6 \times 10^{-2}\,m$

$\ell_z = 2 \times 10^{-2}\,m$

$\ell_x = 6 \times 10^{-3}\,m$

a) $f_1 = \frac{c_y}{4\ell_y}$

$= \frac{45 \times 10^2}{24 \times 10^{-2}}$

$= 1.875 \times 10^4\,HZ$ ✓

b) $E = 10^2$

$W_1 = \frac{\phi^2 E^2}{\rho c S_y}$ ✓

$= \left(\frac{d}{s}\right)^2 \frac{\ell_z E^2}{\rho c \boxed{S_y}}$ $\qquad \left(\frac{d}{s}\right)^2 \frac{\ell_z E^2}{\rho c\, L_x}$

$W_{TOTAL} = 400 \times W_1$

$= 4 \times 10^2 \, (6.2)^2 \, \frac{(2 \times 10^{-2})\, 10^4}{(1.48 \times 10^6)\, 1.2 \times 10^{-4}}$

$= \frac{8 \times 6.2 \times 6.2}{1.48 \times 1.2} \times 10^2$

$= \boxed{1.72 \times 10^4 \text{ WATTS}}$ $\qquad$ 3.45 watts

c) FOR 1 VIBRATOR

$Y = j\omega_1 \frac{\epsilon_x' \epsilon_0 \ell_y \ell_z}{\ell_x} + \frac{\phi^2}{\rho c S_y}$ $\qquad$ if we neglect $\qquad \frac{\phi^2}{Z_m + Z_R}$

$= j\omega_1 \frac{\epsilon_x' \epsilon_0 \ell_y \ell_z}{\ell_x} + \left(\frac{d_{12} \ell_z}{S_{22}}\right)^2 \frac{1}{\rho c S_y}$

$= j\omega_1 \epsilon_x (1 - k^2) \frac{\epsilon_0 \ell_y \ell_z}{\ell_x} + \left(\frac{d_{12} \ell_z}{S_{22}}\right)^2 \frac{1}{\rho c S_y}$

$Im(Y) = 2\pi \times 1.875 \times 10^4 (1 - 0.18^2) \frac{(8.85 \times 10^{-12})(1.2 \times 10^{-5})}{6 \times 10^{-3}}$

$= 2.75\pi \times 10^4 \times 0.968 \times 8.85 \times 10^{-12} \times 1.2 \times 10^{-4} / 6 \times 10^{-3}$

$= (2.75\pi \times 0.968 \times 8.85 \times 2) \times 10^{-9}$

$= \boxed{1.48 \times 10^{-7}}$ $\boxed{\text{SEMENS}}$? $\qquad$ 2.42×10^{-4}

$Re(Y) = \left(\frac{d_{12} \ell_z}{S_{22}}\right)^2 \frac{1}{\rho c S_y}$

$= [(6.2) \times (2 \times 10^{-2})]^2 / (1.48 \times 10^6)(12 \times 10^{-5})$

$= \frac{4 \times 6.2 \times 6.2}{1.48 \times 12} \times 10^{-5}$

$= \frac{6.2 \times 6.2}{1.48 \times 3} \times 10^{-5}$

$= 8.66 \times 10^{-5}$ ✓ $\boxed{\text{SEMENS}}$? $\qquad$ mhos

$Y = 8.66 \times 10^{-5} + j\, 1.48 \times 10^{-7}$

$Y_{TOTAL} = N\,Y \quad \ni \quad N = 400$

9.68×10^{-2}

$Y_{TOTAL} = [3.46 \times 10^{-2} + j\boxed{5.92 \times 10^{-5}}]$ $\boxed{\text{SEMENS}}$? $\qquad$ mhos ?

12.23) $l = 6 \times 10^{-2}$ m

$r_{in} = 5 \times 10^{-3}$ m

$t = 5 \times 10^{-4}$ m

$B_0 = 0.4 \ \frac{\text{WEBER}}{m^2}$

$\mu_i = 80$

$\Lambda = -12 \times 10^{6} \Rightarrow S = 2\pi r t = 2\pi \times 25 \times 10^{-7} = 1.57 \times 10^{-5} \ m^2$

a) $\Lambda = 2 Y K B_0$

$$K = \frac{\Lambda}{2 Y B_0}$$

$$= \frac{-12 \times 10^{6}}{2 \times 21 \times 10^{10} \times 0.4}$$

$$= -7.15 \times 10^{-5} \ \frac{M^4}{\text{WEBER}}$$ ✓

b) $\left(\frac{\delta \xi}{\delta x}\right)_m = K B^2$

$\Rightarrow \frac{\Delta l}{l} = K B^2$

$\Delta l = K B^2 l$

$= (-7.15 \times 10^{-5})(0.16)(6 \times 10^{-2})$

$= 6.85 \times 10^{-7}$ m ✓

c) $F_x = -SY \frac{\delta \xi}{\delta x}$

$= -S Y K B^2$

$= -2\pi r t Y K B^2$

$= -2\pi(5 \times 10^{-3})(5 \times 10^{-4})(21 \times 10^{10})(-7.15 \times 10^{-5})(0.16)$

$= 37.9$ NT ✓

d) $B = B_0 + \mu_i \mu_0 H_i$

$= 0.4 + 80 \times 4\pi \times 10^{-7}\ 4 \times 10^{2}$

$= 0.4 + 1.28\pi \times 10^{-2}$

$= 0.4 + 0.04$

$= .44$

$\Delta l = K B^2 l$

$= -7.15 \times 10^{-5}\ (4.4)^2 \times 10^{-2}\ 6 \times 10^{-2}$

$= -8.30 \times 10^{-7}$

$\Delta(\Delta l) = -(8.30 - 6.85) \times 10^{-7}$

$= -1.75 \times 10^{-7}$ m

e) $F_x = -S Y \frac{\delta \xi}{\delta x}$

$= +1.57 \times 10^{-5} \times 21 \times 10^{10} \times \frac{8.3 \times 10^{-7}}{6 \times 10^{-2}}$

$= 45.6$ NT

$\Delta F = 45.6 - 37.9$

$= 7.7$ NT

PROBLEMS 12.16 & 12.19 WILL HOPEFULLY BE SUBMITTED WITH CHAPT. 13 PROBLEMS ON TUES.

BOB
MARKS

13.1) a) $f = 10^2$ HZ ; $I = 60$ db

FROM FIG 13.10

$LL \approx 36$ PHONS ✓

FROM FIG 13.12 ✓

$L \approx 0.70$ SONES ✓

b) FOR $L = 0.07$ SONES, $LL \approx 17$ PHONES (FROM FIG 13.12)

$\Rightarrow I \approx 48$ db ✓ (FIG 13.10)

c) FOR $L = 7.0$ SONES, ✓ $LL \approx 66$ PHONES (FROM FIG 13.12)

$\Rightarrow I \approx 74$ db ✓ (FIG 13.10)

1.00
.95
1.00
1.00
1.00
1.00
5.95
/6

	FREQ		INTENSITY LEVEL	LOUDNESS LEVEL (FIG 13.12)	
13.2) a)	50 HZ	@	85db	76 PHONS ✓	
	100 HZ	@	80db	76 PHONS ✓	
	200 HZ	@	75db	72 PHONS ✓	
	500 HZ	@	80db	80 PHONS ✓	
	1000 HZ	@	75db	≜ 75 PHONS ✓	
	10,000 HZ	@	70db	72 ^{58} PHONS	c) ΔLOUDNESS LEVEL
b)	50 HZ	@	55db	5 PHONS ✓	✓71 PHONS
	100 HZ	@	50db	18 PHONS ✓	✓58 PHONS
	200 HZ	@	45db	$\overset{30}{26}$ PHONS	✓46 PHONS
	500 HZ	@	30db ^{50}	23 ^{47} PHONS	45 ~~57~~ PHONS
	1000 HZ	@	45db	≜ 45 PHONS	✓30 PHONS
	10,000 HZ	@	40db	28 PHONS	29 ~~44~~ PHONS

13.5)
$$\begin{aligned} r &= a_1 p + a_2 p^2 + a_3 p^3 \\ &= a_1 P \cos \omega t + a_2 P^2 \cos^2 \omega t + a_3 P^3 \cos^3 \omega t \\ &= a_1 P \cos \omega t + \frac{a_2 P^2}{2}\left[1 + \cos 2\omega t\right] \\ &\qquad + \frac{a_3 P^3}{4}\left[3 \cos \omega t + \cos 3\omega t\right] \\ &= \frac{a_2 P^2}{2} + P\left[a_1 + \frac{3a_3}{4} P^2\right] \cos \omega t \\ &\qquad + \frac{a_2 P^2}{2} \cos 2\omega t + \frac{a_3 P^3}{4} \cos 3\omega t \quad \checkmark \end{aligned}$$

13.7) a) 200 HZ IS FROM THE DIFFERENCE OF THE FUNDEMENTAL OF 1200 HZ AND THE SECOND HARMONIC OF 700 HZ.

$2(700) - 1200 = 200$ HZ ✓

b) 300 HZ IS FROM THE DIFFERENCE OF 1200'S SECOND HARMONIC, AND 700'S THIRD HARMONIC.

$2(1200) - 3(700) = 2400 - 2100 = 300$ HZ ✓

c) 1000 HZ IS THE DIFFERENCE OF THE SECOND HARMONICS OF 1200 HZ AND 700 HZ.

$2(1200) - 2(700) = 2400 - 1400 = 1000$ HZ ✓

d) 2200 HZ IS THE DIFFERENCE OF THE THIRD HARMONIC OF 1200 HZ AND THE SECOND OF 700 HZ.

$3(1200) - 2(700) = 3600 - 1400$

$= 2200$ HZ ✓

e) 2600 HZ IS THE SUM OF THE FUNDEMENTAL OF 1200 HZ & THE SECOND HARMONIC OF 700 HZ.

$1200 + 2(700) = 1200 + 1400 = 2600$ HZ ✓

f) 3300 HZ IS THE SUM OF THE FUNDEMENTAL OF 1200 HZ & THE THIRD HARMONIC OF 700 HZ.

$1200 + 3(700) = 1200 + 2100$ ✓

$= 3300$ HZ

13.9) $PSL = SPL - 10 \log_{10} \Delta f$

$= 20 \log_{10} P_e/P_o - 10 \log_{10} \Delta f$

$n_c = -50 db$ Re $\frac{1 \text{ VOLT}}{\text{MICROBAR}} = 20 \log_{10} M_c$

$\Rightarrow 5/2 = \log_{10} 1/M_c$

$\Rightarrow \frac{1}{M_c} = 0.316 \times 10^3$ MICROBARS/VOVT

$M_c = 3.16 \times 10^{-3} \frac{\text{volts}}{\text{MICROBARS}}$

$P_e = V/M_c = 10^{-3}/.316 \times 10^{-2}$

$= 3.16 \times 10^{-1}$ MICROBARS

$P_o = 2 \times 10^{-4}$ MICROBAR

$SPL = 20 \log_{10} P_e/P_o$

$= 20 \log_{10} \frac{3.16 \times 10^{-1}}{2 \times 10^{-4}}$

$= 20 \log_{10} 1.58 \times 10^3$

$= 20(3.2)$

$= 64 db$ Re 2×10^{-4} MICROBAR

$10 \log_{10} \Delta f = 10 \log_{10} 50$

$= 10(1.70)$

$= 17db$

$\therefore PSL = 64 - 17 = 47db$ ✓

13.10) $ISL = 10 \log_{10} \frac{I}{I_0 \Delta f} \ni I_0 = 10^{-12} \frac{WATT}{m^2}$

$I_1 = \frac{10^{-6}}{f} \frac{WATT}{m^2}$

a) i) $f = 10^2 HZ \Rightarrow I_1 = 10^{-8} \frac{WATT}{m^2}$

$ISL = 10 \log_{10} 10^4$

$= 40 db$ Re $10^{-12} \frac{WATT}{m^2}$

ii) $f = 5 \times 10^2 HZ \Rightarrow I_1 = 2 \times 10^{-9} \frac{WATT}{m^2}$

$ISL = 10 \log_{10} (2 \times 10^3)$

$= 33 db$ Re $10^{-12} \frac{WATT}{m^2}$

iii) $f = 10^3 HZ \Rightarrow I_1 = 10^{-9} \frac{WATT}{m^2}$

$ISL = 10 \log_{10} 10^3$

$= 30 db$

b) $IL = ISL + 10 \log_{10} \Delta f$

FOR INTERVAL $10^2_{HZ} < f < 5 \times 10^2 HZ$

$\Delta f = 4 \times 10^2 HZ$; $ISL \approx 36.5 db$

$IL_a = 36.50 + 10 \log_{10} 400$

$= 36.5 + 26.02$

$= 62.5 db$ Re $10^{-12} \frac{WATT}{m^2}$

$= 10 \log_{10} I_a/10^{-12}$

$6.25 = \log_{10} I_a \times 10^{12}$

$-5.75 = \log_{10} I_a$

$5.75 = \log_{10} 1/I_a \Rightarrow \frac{1}{I_a} = 5.68 \times 10^6 \Rightarrow I_a = 1.76 \times 10^{-6} \frac{WATT}{m^2}$

FOR INTERVAL $5 \times 10^2 HZ < f < 10^3 HZ$

$\Delta f = 5 \times 10^2 HZ$; $ISL \simeq 31.5 db$

$IL_b = 31.5 + 10 \log_{10} \Delta f$

$= 31.5 + 10 \log_{10} 5 \times 10^2$

$= 31.5 + 27.0$

$= 58.5 db$ Re $10^{-12} \frac{WATT}{m^2}$

$58.5 = 10 \log_{10} I_b/10^{-12} \Rightarrow 5.85 - 12 = \log_{10} I_b$

$\log_{10} I_b = -6.15$

$1/I_b = 1.41 \times 10^6$

$I_b = 7.07 \times 10^{-7} \frac{WATTS}{m^2}$

$$I_{eq} = I_a + I_b$$
$$= (17.6 + 7.1) \times 10^{-7} \ \frac{\text{WATTS}}{m^2}$$
$$= 24.7 \times 10^{-7} \ \frac{\text{WATT}}{m^2}$$
$$IL_{EQ} = 10 \log_{10} 2.47 \times 10^{6} \quad Re \ 10^{-12} \ \frac{\text{WATT}}{m^2}$$
$$= 10 \, [6.39]$$
$$= 63.9 \text{db} \ Re \ 10^{-12} \ \frac{\text{WATT}}{m^2}$$ ✓

c) FROM FIG. 13.10 & 13.12

$IL_a = 62.5$ db, $f = 350$ HZ $\Rightarrow LL \approx 60$ PHONS $\Rightarrow L = 4.5$ SONES

$IL_b = 58.5$ db, $f = 750$ HZ $\Rightarrow LL = 59$ PHONS $\Rightarrow L \cong 4.0$ SONES

$\Rightarrow$ TOTAL LOUDNESS $\simeq 9.5$ SONES ✓

BOB MARKS

1.00
1.00
.90
1.00
.50
1.00
1.00
1.00

7.40

14.1) a) $I = \frac{W}{a}\left(1 - e^{-\frac{ac}{4V}t}\right)$

$IL = 10 \log_{10} \frac{I}{I_0} \quad \ni \quad I_0 = 10^{-12} \frac{\text{WATTS}}{m^2}$

$= \frac{10}{\log_{10} e} \log_e \frac{I}{I_0}$

$= \frac{10}{\log_{10} e} \ln \frac{W \times 10^{12}}{a}\left(1 - e^{-\frac{ac}{4V}t}\right)$

$\frac{dIL}{dt} = \frac{10}{\log_{10} e} \frac{d}{dt} \ln \frac{W \times 10^{12}}{a}\left(1 - e^{-\frac{ac}{4V}t}\right)$

$= \frac{10}{\log_{10} e} \frac{d}{dt}\left[\ln \frac{W \times 10^{12}}{a} + \ln\left(1 - e^{-\frac{ac}{4V}t}\right)\right]$

$= \frac{10}{\log_{10} e} \frac{d}{dt} \ln\left(1 - e^{-\frac{ac}{4V}t}\right)$

$= \frac{10}{\log_{10} e} \frac{1}{\left(1 - e^{-\frac{ac}{4V}t}\right)} \frac{d}{dt}\left(1 - e^{-\frac{ac}{4V}t}\right)$

$= \frac{10}{\log_{10} e}\left[+\frac{ac}{4V} \frac{e^{-\frac{ac}{4V}t}}{1 - e^{-\frac{ac}{4V}t}}\right]$

$= \frac{2.5}{\log_{10} e} \frac{ac}{V} \frac{1}{e^{\frac{ac}{4V}t} - 1}$

$= \frac{2.5}{3.303} \frac{ac}{V} \frac{1}{e^{\frac{ac}{4V}t} - 1}$

$= 1.087 \frac{ac}{V} \frac{1}{e^{\frac{ac}{4V}t} - 1}$ ✓

b) $t = 0 \Rightarrow \frac{dIL}{dt} = \infty \quad \text{SINCE} \quad \left(e^{\frac{ac}{4V}t} - 1\right)\Big|_{t=0} = 0$ ✓

$t = \infty \Rightarrow \frac{dIL}{dt} = 0 \quad \text{SINCE} \quad \left(e^{\frac{ac}{4V}t} - 1\right)\Big|_{t=\infty} = \infty$ ✓

c) $D = \frac{dIL}{dt}$

$\frac{1.087ac}{V} = \frac{1.087\,ac}{V} \frac{1}{e^{\frac{ac}{4V}t} - 1}$

$\Rightarrow e^{\frac{ac}{4V}t} = 2$

$\frac{ac}{4V} t = \ln 2$

$t = \frac{4V}{ac} \ln 2$

$= \frac{V}{ac} \times 2.76$ ✓

14.3) $l = 6$ FT

$w = 7$ FT

$h = 8$ FT

a) $f = 2 \times 10^3$ HZ

$\alpha = 0.02$

$W = 7.5 \times 10^{-6}$

$$P_\infty^2 = \frac{4W\rho_0 c}{a'}$$

$$a_{MKS} = \Sigma \alpha_i S_i$$

$$= \alpha \Sigma S_i$$

$$= \alpha(2\times6\times7 + 2\times6\times8 + 2\times7\times8)\,\text{FT}^2 \times \frac{9.29\times10^{-2}\,m^2}{\text{FT}^2}$$

$$=(2\times10^{-2})(9.29\times10^{-2})(84+96+112)$$

$$=(18.58\times10^{-4})(2.92\times10^{2})$$

$$=54.2\times10^{-2}$$

$$=0.542$$

$$P_\infty^2 = \frac{4(7.5\times10^{-6})(4.15\times10^{2})}{0.542}$$

$$=250\times10^{-4}$$

$$=2.5\times10^{-2}$$

$$\Rightarrow P_\infty = 0.158\,\frac{NT}{M^2}$$

$$SPL_\infty = 20\log_{10} P_\infty/P_0$$

$$= 20\log_{10}\frac{1.58\text{ MICROBARS}}{2\times10^{-4}\text{ MICROBARS}}$$

$$= 20\log_{10} 7.95\times10^{3}$$

$$= 20(3.90)$$

$$= 78\text{ db}$$

✓

b) $SPL_\infty = 20 \log_{10} \frac{P_\infty}{P_0}$

$$SPL = SPL_\infty - 3 = 20 \log_{10} P/P_0 = 20 \log_{10} \frac{P_\infty}{P_0} - 3$$

$$10 \log_{10} \left(\frac{P_\infty}{P_0}\right)^2 - 3 = 10 \log_{10} (P/P_0)^2$$

$$10 \log_{10} \frac{4W\rho_0 c}{P_0^2 a} - 3 = 10 \log_{10} \frac{4W\rho_0 c}{a P_0^2} \left(1 - e^{-\frac{ac}{4V} t}\right)$$

$$10 \log_{10} \frac{4W\rho_0 c}{P_0^2 a} - 3 = 10 \log_{10} \frac{4W\rho_0 c}{a P_0^2} + 10 \log_{10} \left(1 - e^{-\frac{ac}{4V} t}\right)$$

$$-3 = 10 \log_{10} \left(1 - e^{-\frac{ac}{4V} t}\right)$$

$$10^{-0.3} = 1 - e^{-\frac{ac}{4V} t}$$

$$e^{-\frac{ac}{4V} t} = 1 - 10^{-0.3}$$

$$-\frac{ac}{4V} t = \ln\left(1 - 10^{-0.3}\right)$$

$$t = \frac{-4V}{ac} \ln\left(1 - 10^{-0.3}\right)$$

$$10^{0.3} = 2.00$$

$$\therefore t = \frac{-4V}{ac} \ln(0.5)$$

$$= \frac{4V}{ac} \ln(2.0)$$

$$= \frac{4 \times 3.36 \times 10^2 \times 2.83 \times 10^{-2} \times 0.693}{(0.542)(3.43 \times 10^2)}$$

$$= 1.41 \times 10^{\boxed{-2}} \text{ SEC}$$

10^{-1}

c) ASSUME $t = \infty$

$SPL_1 = 10 \log_{10} \left(\frac{P_1}{P_0}\right)^2$; $SPL_2 = 10 \log_{10} \left(\frac{P_2}{P_0}\right)^2$

$P_1{}^2 = \frac{4W\rho_0 c}{a_1}$ $P_2{}^2 = \frac{4W\rho_0 c}{a_2}$

$a_1 = (2 \times 10^{-2})(2.92 \times 10^2)$

$= 5.84$ SABINS

$SPL_1 - SPL_2 = 3$

$10 \log_{10} \left(\frac{P_1}{P_0}\right)^2 - 10 \log_{10} \left(\frac{P_2}{P_0}\right)^2 = 3$

$10 \log_{10} \frac{1}{P_0^2} \frac{4W\rho_0 c}{a_1} - 10 \log_{10} \frac{1}{P_0^2} \frac{4W\rho_0 c}{a_2} = 3$

$10 \log_{10} \frac{4W\rho_0 c}{P_0^2} - 10 \log_{10} a_1 - \log_{10} \frac{4W\rho_0 c}{P_0^2} + 10 \log_{10} a_2 = 3$

$10 \log_{10} a_2/a_1 = 3$

$\log_{10} a_2/a_1 = 0.3$

$a_2/a_1 = 10^{0.3} = 2.00$

$a_2 = 2a_1$

$\Rightarrow \Delta a = a_2 - a_1 = a_1 = 5.84$ SABIN

14.4) $l = 10\,m \quad w = 10\,m, \quad h = 4.0\,m$

$\bar{\alpha} = 0.1 \ \frac{SABINS}{ft^2}$

a) $a = 0.1 \ \frac{SABINS}{FT^2} \times 2(10\times10 + 10\times4 + 10\times4)\,m^2$

$= 0.1 \times 360 \ \frac{SABIN\ m^2}{FT^2}$

$= 36 \times \frac{10.76\ FT^2}{m^2} \ \frac{SABIN\ m^2}{FT^2}$

$= 388$ SABIN

$V = 400\,M^3 \times \frac{35.3\ ft^3}{m^3}$

$= 1.41 \times 10^4\ ft^3$

$T = \frac{(4.9\times10^{-2})(1.41\times10^4)}{3.88\times10^2}$

$= 1.78$ SEC ✓

b) SPL = 60db R_e 2×10^{-4} MICROBAR $= P_0$

$SPL = 10\log_{10}(P/P_0)^2$

$P_\infty^2 = \frac{4W\rho_0 c}{a}$

$SPL = 60 = 10\log_{10}\frac{4W\rho_0 c}{P_0^2\, a}$

$1/6 = \log_{10}\frac{4W\rho_0 c}{P_0^2\, a}$

$\frac{4W\rho_0 c}{P_0^2\, a} = \boxed{10^{1/6}} \quad 10^6$

$W = \frac{P_0^2\, a}{4\rho_0 c}\, 10^{1/6}$

$a_{MKS} = 3.88\times10^2 \times \frac{1\ m^2}{10.76\ ft^2}$

$= 36.0$ ✓

$P_0^2 = (2\times10^{-4}\ MICROBAR \times \frac{10^{-1}\ NT/M^2}{MICROBAR})^2$

$= 4\times\boxed{10^{-6}}\ (\frac{NT}{m^2})^2 \qquad 4\times10^{-10}$

$W = \frac{(4\times10^{-6})(36.1)\boxed{(1.47)}}{4(4.15\times10^2)}\ 10^6$

~~$= 12.8\times10^{-8}$~~ $\qquad$ 8.67 microwatts

~~$= 0.128\ \mu WATTS$~~

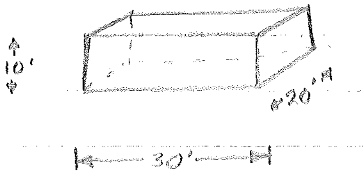
c) $I = \frac{W}{a_{MKS}}$

$=$ ~~$\frac{12.8\times10^{-9}}{36.1}$~~ $\qquad 8.67\times10^{-6}$

$=$ ~~3.54×10^{-9}~~ $\frac{WATTS}{m^2}$

.241 microwatts/meter

14.6) $h = 10'$ $w = 20'$ $\ell = 30'$



WALLS: $\bar{\alpha} = 0.05$

FLOOR: $\alpha = 0.2$

CIELING: $\alpha = 0.6$

PEOPLE: $a_p = 4.5$ SABINS

a) $T = \frac{(4.9\times10^{-2})(V)}{a}$

$a = \Sigma \alpha_i S_i$

$a_{WALLS} = 0.05\times2\,(10\times20 + 10\times30)$

$= 50$ SABINS

$a_{FLOOR} = 0.2\times30\times20$

$= 120$

$a_{CIELING} = 0.6\times30\times20$; $a_{PEOPLE} = 10\times4.5 = 45$

$= 360$

$\therefore a = 50+120+360+45 = 575$ SABINS

$V = 10\times20\times30 = 6\times10^3\ ft^3$

$T = \frac{(4.9\times10^{-2})(6\times10^3)}{5.75\times10^2}$

$= 0.512$ SEC ✓

b) $T = \dfrac{4.9 \times 10^{-2} V}{-S \ln(1-\bar{\alpha})}$

$S = 2(10 \times 20 + 10 \times 30 + 20 \times 30)$

$= 2.20 \times 10^3 \text{ ft}^2$

$\bar{\alpha} = \dfrac{1}{S} \sum S_i \alpha_i$

$= \dfrac{a_a}{S}$ $\ni a_a$ AS COMPUTED IN PART a

$= \dfrac{5.75 \times 10^2}{2.20 \times 10^3}$

$= 0.261$

$\ln(1-\bar{\alpha}) = \ln(0.739)$

$= -\ln(1.354)$

$= -0.303$

$\Rightarrow T = \dfrac{(4.9 \times 10^{-2})(6 \times 10^3)}{(2.2 \times 10^3)(0.303)}$

$= 0.442$ SEC ✓

c) $T = \dfrac{4.9\times10^{-2}\,V}{\Sigma - S_i \ln(1-\alpha_i)}$

$a = \Sigma - S_i \ln(1-\alpha_i)$

$a_{\text{WALLS}} = -10^3 \ln(1-0.05)$

$= -10^{+3} \ln(0.95)$

$= 10^3 \ln 1.0526$

$= 0.0513 \times 10^3$

$= 51.3$ ✓

$a_{\text{FLOOR}} = -6\times10^2 \ln(0.8)$

$= 6\times10^2 (\ln 1.25)$

$= 6\times10^2 (0.223)$

$= 1.34\times10^2$

$= 134$ ✓

$a_{\text{CIELING}} = -6\times10^2 \ln(0.4)$

$= +6\times10^2 \ln(2.5)$

$= 6\times10^2 (0.916)$

$= 55.\underset{\curvearrowright}{0}.$

$a_{\text{PEOPLE}} = 45.0$ ✓

$\Sigma a = \cancel{285}\ 779$

$$\Rightarrow T = \frac{4.9\times10^{-2}\times6\times10^{3}}{\begin{array}{c}\cancel{2.85\times10^{2}}\\ 779\end{array}} = .38$$

~~= 1.03 SEC~~

α_i = coef listed in table

α_e = effective coef in calc. rev. time for dead room

$\alpha_e = -\ln(1-\alpha_i)$

For acoustic panelling at 125 hertz

$\alpha_e = -\ln(1-.16) = .17$

For carpet

$\alpha_e = -\ln(1-.11) = .116$

14.8) $w = h = l = 10' \Rightarrow V = 10^3$

$T = \frac{4.9 \times 10^{-2} V}{a} \ni a = \Sigma \alpha_i S_i$

$\alpha_e = -\ln(1-\alpha_i)$??

$e^{\alpha e} = -1 + \alpha_i$

$\alpha_i = e^{\alpha e} + 1$

	S_i	500 HZ		125 HZ		2000 HZ	
		α_e	$e^{\alpha_e}+1$	α_e	$e^{\alpha_e}+1$	α_e	$e^{\alpha_e}+1$
ACOUSTIC PLASTER	10^2	.0.50	2.65	0.30	2.35	0.55	2.74
ACOUSTIC PANELING	4×10^2	0.50	2.65	0.16	2.17	0.80	3.23
CARPETING	10^2	0.37	2.45	0.11	2.12	0.27	2.31

a) $f = 500$ HZ

$a = (2.65 + 4 \times 2.65 + 2.45) \times 10^2$

$= 15.7 \times 10^2$ SABIN

$T_{500} = \frac{(4.9 \times 10^{-2})(10^3)}{1.5 \times 10^3} = 3.26 \times 10^{-2}$ SEC

?

b) $f = 250$ HZ

$a = (2.35 + 4 \times 2.17 + 2.12) \times 10^2$

$= (4.47 + 8.68) \times 10^2 = 13.15 \times 10^2$

$T_{250} = \frac{(4.9 \times 10^{-2}) \times 10^3}{1.315 \times 10^3} = 3.73 \times 10^{-2}$ SEC

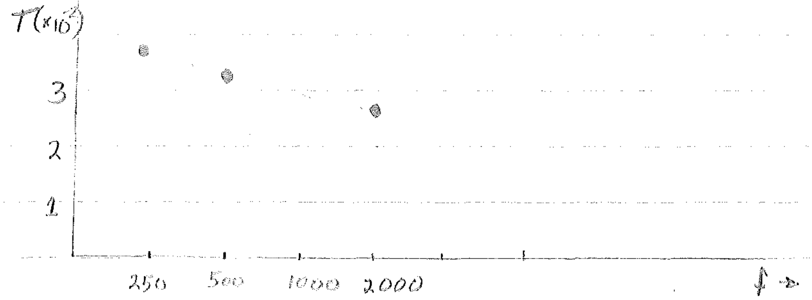
c) $f = 2000$ HZ

$a = (2.74 + 4 \times 3.23 + 2.31) \times 10^2$

$= (5.05 + 12.92) \times 10^2 = 17.97 \times 10^2$

$a = \sum S_i \alpha_{e_i}$

$T_{2000} = \frac{(4.9 \times 10^{-2}) \times 10^3}{1.8 \times 10^3} = 2.72 \times 10^{-2}$ SEC



14.14) a) $T = \dfrac{4.9\times10^{-2}V}{a+4mV}$; $f = 6\times10^{3}\,\mathrm{HZ}$

$$\frac{1}{T} = \frac{a+4mV}{4.9\times10^{-2}V}$$

$$= \frac{a}{4.9\times10^{-2}V} + \frac{4m}{4.9\times10^{-2}}$$

$$\Rightarrow \frac{1}{T} - \frac{4m}{4.9\times10^{-2}} = \frac{a}{4.9\times10^{-2}V}$$

AND $\dfrac{1}{T'} - \dfrac{4m'}{4.9\times10^{-2}} = \dfrac{a}{4.9\times10^{-2}V}$

COMBINING:

$$\frac{1}{T} - \frac{4m}{4.9\times10^{-2}} = \frac{1}{T'} - \frac{4m'}{4.9\times10^{-2}}$$

$$\frac{4m'}{4.9\times10^{-2}} = \left(\frac{1}{T'} - \frac{1}{T}\right) + \frac{4m}{4.9\times10^{-2}}$$

$$m' = \frac{4.9\times10^{-2}}{4}\left(\frac{1}{T'} - \frac{1}{T}\right) + m$$ ✓

b) FROM TABLE 9-1

$\dfrac{\alpha}{f^2} = 2.0\times10^{-11}$ FOR DRY AIR @ 20°C AND 1 ATM ✓

$$\alpha = f^2(2.0\times10^{-11})$$

$$= (6\times10^{3})^2(2.0\times10^{-11})\ \text{NEPER/m}$$

$$= 72\times10^{-5}\ \text{NEPER/m}$$

$$m = 2\alpha = 1.44\times10^{-3}\ /\text{m}$$

$$= 1.44\times10^{-3}\ /\text{m} \times \frac{0.305\ \text{m}}{\text{ft}}$$

$$= 0.439\times10^{-3} = 4.39\times10^{-4}\ /\text{FT}$$ ✓

$$m' = \frac{4.9\times10^{-2}}{4}\left(\frac{1}{5} - \frac{1}{20}\right) + 4.39\times10^{-4}$$

$$= 1.225\times10^{-2}(0.20 - 0.05) + 4.39\times10^{-4}$$

$$= 1.225\times10^{-2}\times0.15 + 4.39\times10^{-4}$$

$$= 1.84\times10^{-3} + 4.93\times10^{-4}$$

$$= 2.33\times10^{-3}\ /\text{FT}$$ ✓

14.19) a) $SPL = 74 \text{ db Re } 2 \times 10^{-4}$ MICROBAR

$= 20 \log_{10} P_e/P_0$

$\log_{10} P_e/P_0 = 3.7$

$P_e/P_0 = 5 \times 10^3$

$P_e = (5 \times 10^3)(2 \times 10^{-4})$

$= 1$ MICROBAR

$W = 2.8 \times 10^{-8} \frac{P^2 V}{T}$

$= 2.8 \times 10^{-8} \frac{(1)^2 (10^4)}{2}$

$= 1.40 \times 10^{-4}$ WATTS ✓

b) $SPL' = 64 \text{db Re } 2 \times 10^{-4}$ MICROBARS

$= 20 \log_{10} P_e'/P_0$

$3.2 = \log_{10} \frac{P_e'}{P_0} \Rightarrow P_e'/P_0 = 1.585 \times 10^3$

$P_e' = 3.17 \times 10^{-1}$ MICROBARS

$P_e'^2 = \frac{4 W \rho_0 c}{a'}$; $P_e^2 = \frac{4 W \rho_0 c}{a}$

$\Rightarrow a' P_e'^2 = a P_e^2$

$a' = a \left(\frac{P_e^2}{P_e'^2} \right)$

NOW $W = \frac{P^2 a}{4 \rho_0 c} \Rightarrow a = \frac{4 \rho_0 c W}{P^2}$

$\Delta a = (a' - a) = \left(\frac{P_e}{P_e'} \right)^2 a - a$ (METRIC SABINS)

$= a \left(\frac{P_e^2}{P_e'^2} - 1 \right)$

$= \frac{4 \rho_0 c W}{P_e^2} \left(\frac{P_e^2}{P_e'^2} - 1 \right)$

$= 4 \rho_0 c W \left(\frac{1}{P_e'^2} - \frac{1}{P_e^2} \right)$

$= 4 \times 4.15 \times 10^2 \times 1.40 \times 10^{-4} \left(\frac{1}{3.17^2 \times 10^{-4}} - \frac{1}{10^{-2}} \right)$

$= 23.3 \times 10^{-2} (10^3 - 10^2)$

$= 9.0 \times 10^2 \times .233$

$= 2.1 \times 10^{+2}$ METRIC SABINS ✓

c) $W = 2.8 \times \frac{P^2 V}{T} \times 10^{-8}$

$\Rightarrow T = 2.8 \times 10^{-8} \frac{P^2 V}{W}$

$= 2.8 \times 10^{-8} \frac{(3.17)^2 \times 10^{-2} \times 10^{4}}{1.4 \times 10^{-4}}$

$= 2.0 \times 10^{-1}$

$= 0.2$ SEC ✓

14.21) a) $\Delta f = 1$ HZ

$V = 2 \times 3 \times 10 = 60\ m^3$

$S = 2(2\times3 + 2\times10 + 3\times10)$

$= 2(6+20+30) = 2\times56$

$= 112\ m^2$; $L = 4(2+3+10) = 60\ m$

$$\Delta N = \left(\frac{4\pi V}{c^3} f^2 + \frac{\pi S}{2c^2} f + \frac{L}{8c}\right)\Delta f$$

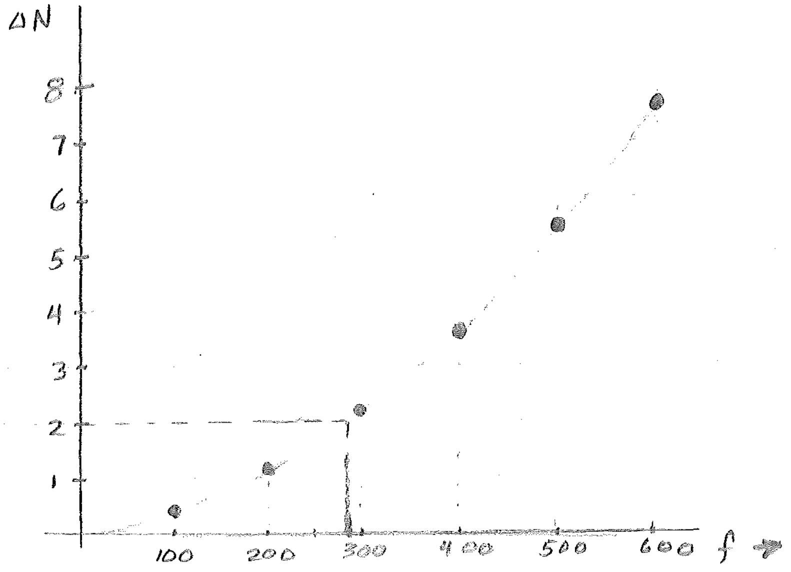
$C = 343 \frac{m}{SEC}$

$$\frac{4\pi V}{c^3} = \frac{4\times\pi\times60}{(3.43)^3\times10^6} = 1.87\times10^{-5}$$

$$\frac{\pi S}{2c^2} = \frac{\pi\times1.12\times10^2}{2(3.43)^2\times10^4} = 1.5\times10^{-3}$$

$$\frac{L}{8c} = \frac{60}{8\times3.43\times10^2} = 2.19\times10^{-2}$$

f	f^2	$\overset{1.87\times10^{-5}}{\overbrace{\frac{4\pi V}{c^3}}} f^2$	$\overset{1.5\times10^{-3}}{\overbrace{\frac{\pi S}{2c^2}}} f$	$\frac{L}{8c}$	ΔN
10^2	10^4	1.87×10^{-1}	1.5×10^{-1}	0.22×10^{-1}	0.36
2×10^2	4×10^4	7.43×10^{-1}	3.0×10^{-1}	0.22×10^{-1}	1.07
3×10^2	9×10^4	17.00×10^{-1}	4.5×10^{-1}	0.22×10^{-1}	2.17
4×10^2	1.6×10^5	29.9×10^{-1}	6.0×10^{-1}	0.22×10^{-1}	3.61
5×10^2	2.5×10^5	46.7×10^{-1}	7.5×10^{-1}	0.22×10^{-1}	5.44
6×10^2	3.6×10^5	67.3×10^{-1}	9.0×10^{-1}	0.22×10^{-1}	7.66



✓

b) FROM GRAPH, $\Delta N = 2$ @ $f \doteq 290$ HZ